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# Geometry of quantum systems: density states and entanglement

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## Abstract

Various problems concerning the geometry of the space  $u^*(\mathcal{H})$  of Hermitian operators on a Hilbert space  $\mathcal{H}$  are addressed. In particular, we study the canonical Poisson and Riemann–Jordan tensors and the corresponding foliations into Kähler submanifolds. It is also shown that the space  $\mathcal{D}(\mathcal{H})$  of density states on an  $n$ -dimensional Hilbert space  $\mathcal{H}$  is naturally a manifold stratified space with the stratification induced by the rank of the state. Thus the space  $\mathcal{D}^k(\mathcal{H})$  of rank- $k$  states,  $k = 1, \dots, n$ , is a smooth manifold of (real) dimension  $2nk - k^2 - 1$  and this stratification is maximal in the sense that every smooth curve in  $\mathcal{D}(\mathcal{H})$ , viewed as a subset of the dual  $u^*(\mathcal{H})$  to the Lie algebra of the unitary group  $U(\mathcal{H})$ , at every point must be tangent to the strata  $\mathcal{D}^k(\mathcal{H})$  it crosses. For a quantum composite system, i.e. for a Hilbert space decomposition  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ , an abstract criterion of entanglement is proved.

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## 1. Introduction

Dirac's approach to quantum mechanics uses a Hilbert space as a fundamental object to start with, motivating the linear structure with the superposition principle necessary to describe phenomena like those of interference [1]. Born's probabilistic interpretation requires the use of a Hermitian inner product to deal with normalized states; therefore the physical identification of states in the Hilbert space leads to the requirement that (pure) states of a quantum mechanical system are described by elements of the complex projective space

(one-dimensional subspaces of a separable complex Hilbert space  $\mathcal{H}$ ). By means of the Hermitian structure on  $\mathcal{H}$  it is possible to define a binary product on the pure states  $P\mathcal{H}$  [2, 4, 5]. The physical interpretation of this binary operation is given in terms of probability transition from one state to another. On this space  $P\mathcal{H}$ , bijective maps which preserve the transition probability are necessary projection of unitary or anti-unitary transformations on the original Hilbert space; this statement is the main content of Wigner's theorem [6].

More likely, due to this 'equivalence' between the two descriptions (on  $\mathcal{H}$  and on  $P\mathcal{H}$ ), physicists have barely paid any attention to the geometrization of quantum mechanics; i.e. to introduce a 'tensorial description' in such a way that nonlinear coordinate transformations could be performed; notable exceptions obviously do exist and we provide a partial list of references [12]. The recent great interest in the foundational aspects of quantum mechanics motivated by the use of entanglement as a resource for quantum information and quantum computing has boosted a deeper study of many fundamental aspects, for instance the possibility of having a binary composition of pure states without the use of the Hilbert space linear structure [8, 9], the possibility of having a nonlinear quantum mechanics [9, 11], more generally the possibility of having nonlinear transformations among states.

The possibility of nonlinear transformations may turn out to be quite useful in the classification problem of separability and entanglement because these properties are not preserved by taking linear combinations. Moreover, an appropriate description of atomic phenomena, involving polarization, spin orientation and angular correlations, requires that we go beyond pure states in the description of quantum systems. This larger family of states was introduced by von Neumann as dual objects with respect to the quantum observables; they constitute the set of density states and an early, physically motivated, review was written by U Fano [13].

Again, for these states a proper mathematical setting is provided by the dual space of the Lie algebra of the observables, with respect to the coadjoint action of the unitary group. Density states emerge as elements of the coadjoint orbits passing through some special elements in the dual of the Cartan subalgebra. The mathematical context of coadjoint orbits is quite well known to those physicists involved with geometric quantization and the field was widely studied in the 1970s by Kostant, Kirillov and Souriau [14].

Each coadjoint orbit bears a natural differential structure. Observe, however, that the spectrum of the state does not change along the orbit of the unitary action. From the point of view of quantum evolution it corresponds to the situation of an isolated system, when all interactions with the environment are negligible, so there is no dissipation and the evolution is unitary. In many cases this is only a very exceptional situation, very rarely adequately corresponding to the physical reality. On the other hand, it is *a priori* not clear that the whole set of density states, i.e. a union of coadjoint orbits of the unitary action of different dimensionalities, possesses a natural differential structure. Exhibiting such a structure in terms of local coordinates and/or *via* a general geometric construction of a smooth stratification of density states is thus of great interest when investigating dissipative systems.

Density states form a convex subset of the set of Hermitian operators on  $\mathcal{H}$ . Some properties of these convex bodies attracted recently attention [15]. It is thus legitimate to ask about 'the shape' of the set of density matrices, in particular about the smoothness properties of its boundary. In the simplest case of the two-dimensional  $\mathcal{H}$ , the density matrices form the three-dimensional unit ball with a smooth boundary—the two-dimensional unit sphere comprising all pure states. But this situation is exceptional—in higher dimensions the boundary does not consist exclusively of pure states; it is in addition not smooth.

The space of density states carries additional structures with respect to those available on the space of coadjoint orbits of general Lie groups because they are related to the unitary group

and therefore additional structures are available. Moreover, the need to consider composite quantum systems, tensor products of the spaces associated with a choice of subsystems making up the whole system, will bring up novel problems which will require further investigations.

All these various considerations have convinced us that a review of these mathematical aspects along with the identification of the novel emerging problems may be useful to those interested in the application of quantum mechanics to quantum information and are not familiar with the geometrical background required. A recent book by Chruściński and Jamiołkowski [16] deals with geometrical aspects of quantum mechanics; these authors however are primarily concerned with the application of these methods to describe the geometric phase [17]. At this point one should also refer to the paper [18] in which, in connection with geometric phase and parallel transport along mixed states, the geometry of the manifold of density matrices as a stratified space was discussed along slightly different lines than in the present paper (cf section 3 below).

In the present paper the Hilbert space  $\mathcal{H}$  will be assumed to be of finite dimension  $n$  in order to make the differential geometry expressible in local coordinates classical. The reader will understand that passing to an infinite-dimensional  $\mathcal{H}$  (i.e. a differential geometry of a Banach (or a Hilbert) manifold) is straightforward; to this aim we will try to use coordinate-free expressions, which serve in both cases, as much as possible. The paper is organized as follows.

In section 2 we start with presenting the Kähler structure on the Hilbert projective space  $P\mathcal{H}$  of pure states obtained from the standard Hermitian product on  $\mathcal{H}$  via the momentum map associated with the Hamiltonian action of the group  $U(\mathcal{H})$  of unitary transformations of  $\mathcal{H}$ . In this picture the pure states form just an orbit in the dual space  $u^*(\mathcal{H})$  of the unitary Lie algebra  $u(\mathcal{H})$  of the group  $U(\mathcal{H})$ . Because of the nondegeneracy of the canonical invariant scalar product on  $u(\mathcal{H})$  we have a canonical identification of  $u^*(\mathcal{H})$  with  $u(\mathcal{H})$  which makes the geometry of  $u^*(\mathcal{H})$  very rich. We decided to interpret  $u^*(\mathcal{H})$  as the space of Hermitian operators on  $\mathcal{H}$  which makes possible to understand the density states as a subset of  $u^*(\mathcal{H})$ .

Consequently, in sections 3 and 4 we present the density states as a convex body  $\mathcal{D}(\mathcal{H})$  in  $u^*(\mathcal{H})$  which is a family of some  $U(\mathcal{H})$ -orbits and, as we will show later, also orbits of a particular action of the group  $GL(\mathcal{H})$  of invertible complex linear operators on  $\mathcal{H}$ . We show that  $\mathcal{D}(\mathcal{H})$  is naturally a manifold stratified space with the stratification induced by the rank of the state. Thus the space  $\mathcal{D}^k(\mathcal{H})$  of rank- $k$  states,  $k = 1, \dots, n$ , is a smooth manifold of (real) dimension  $2nk - k^2 - 1$  and this stratification is maximal in the sense that every smooth curve in  $\mathcal{D}(\mathcal{H})$ , viewed as a subset of the dual  $u^*(\mathcal{H})$  to the Lie algebra of the unitary group  $U(\mathcal{H})$ , at every point must be tangent to the strata  $\mathcal{D}^k(\mathcal{H})$  it crosses.

Section 5 is devoted to the geometry of  $u^*(\mathcal{H})$ , to a global description of the Kählerian structure of  $U(\mathcal{H})$ -orbits by means of the canonical Poisson and Riemann–Jordan tensors. These Kählerian structures are well known in algebraic geometry and can be easily generalized to analogous structures on general flag manifolds. The point which should be stressed here is that the geometry we develop is canonical in that it does not depend on the matrix form of an operator and the  $U(\mathcal{H})$ -orbits are treated as a collection rather than each orbit separately.

In the last section we investigate a Hilbert space decomposition  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$  which is usually understood as corresponding to a quantum composite system. We present an introduction to the problems of separability and entanglement together with an abstract scheme for measurement of entanglement.

The geometry of composite quantum systems was investigated in the literature from several points of view. First, it is of importance to distinguish classes of states which are equivalent under a restricted set of unitary transformations (dubbed local transformations in the physical literature), namely those which belong to the same orbit of  $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$ . From

the physical point of view all states on the same orbit contain an equal amount of quantum correlations between the subsystems; i.e., these cannot be influenced by operations performed separately on each subsystem.

In order to characterize uniquely an orbit (i.e. a class of locally equivalent states) one can try to find a complete set of  $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$ -invariant functions on  $\mathcal{D}(\mathcal{H})$  such that the values of all functions at  $\rho \in \mathcal{D}(\mathcal{H})$  characterize uniquely the orbit through  $\rho$  [19, 20]. The task can be effectively completed only for low-dimensional systems—in fact, only in the case  $\dim \mathcal{H}^1 = \dim \mathcal{H}^2 = 2$  the explicit results were found [21]. The same is true for multipartite composite systems, i.e. when  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \dots \otimes \mathcal{H}^K$ . Here also the explicit results are known for  $K$  up to 4 and  $\dim \mathcal{H}^i = 2, i = 1, \dots, 4$  [22, 23].

Other (partial) characterization of local orbits is provided by their dimensions. These were investigated in [24, 25] and in [26] all orbits of submaximal dimensionality in the case  $\dim \mathcal{H}^1 = \dim \mathcal{H}^2 = 2$  were explicitly identified and enumerated. A similar task of finding dimension of the local orbit through an arbitrary  $\rho$  in the case of higher dimensional systems was never achieved. A much more modest goal of determining dimensions and topology of local orbits stratifying the set of rank 1 (pure) states  $\mathcal{D}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$  was, however, completed for arbitrary finite-dimensional  $\mathcal{H}^1$  and  $\mathcal{H}^2$  [27].

The sets of pure states in two- and three-partite systems with  $\dim \mathcal{H}^i = 2$  can be identified with, respectively, unit seven- and fifteen-dimensional spheres  $\mathbf{S}^7$  and  $\mathbf{S}^{15}$ . In both cases there exist the Hopf fibrations  $\mathbf{S}^7 \rightarrow \mathbf{S}^4$  and  $\mathbf{S}^{15} \rightarrow \mathbf{S}^8$  which were used to investigate the geometry of pure states in [28–31], whereas multipartite pure states were treated in [32] using Segre variety.

Although in the present paper we limit ourselves to investigation of two-partite composite system, we would like to point out recent achievements in geometric characterization of entangled pure states of multipartite systems. When investigating entanglement in multicomponent system one aims at discriminating among different classes of entanglement, defined as different equivalence classes under appropriate group of transformations preserving entanglement properties. The goal can be achieved by identification of the so-called entanglement monotones, i.e. measures of entanglement which are invariant under considered transformations. Construction of such invariants based on Plücker coordinates on Grassmannians, naturally appearing when considering pure states of multicomponent systems, were presented in [33] and [34]. The geometry of three-qubit pure state entanglement was recently investigated in [35], where the geometric description of different classes of entanglement was given in terms of submanifolds of the so-called Klein quadric—a special quadric embedded in the five-dimensional complex projective space.

Of special interest is the set of separable states (defined in section 6), as those which, from the physical point of view, do not carry any quantum correlations. From the construction they form a convex subset in  $\mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Only in the case of  $\dim \mathcal{H}^1 = \dim \mathcal{H}^2 = 2$  and  $\dim \mathcal{H}^1 = 2$  and  $\dim \mathcal{H}^2 = 3$  (or vice versa) do there exist effective criteria which allow us to discriminate separable and nonseparable (entangled) states. As a consequence only in these low-dimensional cases can one relatively easily investigate the geometry of the boundary of the set of separable states [37].

## 2. Kähler structure on the Hilbert projective space

Let  $\mathcal{H}$  be an  $n$ -dimensional Hilbert space with the Hermitian product  $\langle x, y \rangle_{\mathcal{H}}$  being, by convention,  $\mathbf{C}$ -linear with respect to  $y$  and anti-linear with respect to  $x$ . The unitary group  $U(\mathcal{H})$  acts on  $\mathcal{H}$  preserving the Hermitian product and it consists of those complex linear

operators  $A \in gl(\mathcal{H})$  on  $\mathcal{H}$  which satisfy  $AA^\dagger = I$ , where  $A^\dagger$  is the Hermitian conjugate of  $A$ , i.e.

$$\langle Ax, y \rangle_{\mathcal{H}} = \langle x, A^\dagger y \rangle_{\mathcal{H}}.$$

The geometric approach to quantum mechanics is based on considering the realification  $\mathcal{H}_{\mathbf{R}}$  of  $\mathcal{H}$  as a Kähler manifold  $(\mathcal{H}_{\mathbf{R}}, J, g, \omega)$  with canonical structures: a complex structure  $J : T\mathcal{H}_{\mathbf{R}} \rightarrow T\mathcal{H}_{\mathbf{R}}$ , a Riemannian metric  $g$  and a symplectic form  $\omega$ . The latter come from the real and imaginary parts of the Hermitian product, respectively,  $g = \Re(\langle \cdot, \cdot \rangle_{\mathcal{H}})$ ,  $\omega = \Im(\langle \cdot, \cdot \rangle_{\mathcal{H}})$ . After the obvious identification of the tangent bundle  $T\mathcal{H}_{\mathbf{R}}$  with  $\mathcal{H}_{\mathbf{R}} \times \mathcal{H}_{\mathbf{R}}$ , all these structures are constant structures induced from  $\mathcal{H}$ :

$$J(x) = i \cdot x, \quad g(x, y) + i \cdot \omega(x, y) = \langle x, y \rangle_{\mathcal{H}}.$$

We have obvious identities

$$J^2 = -I, \quad \omega(x, Jy) = g(x, y), \quad g(Jx, Jy) = g(x, y), \quad \omega(Jx, Jy) = \omega(x, y).$$

The tensors  $g$  and  $\omega$  being non-degenerate have their inverses: the contravariant metric tensor  $G = g^{-1}$  and the Poisson tensor  $\Omega = \omega^{-1}$ . They form together a Hermitian product

$$\langle \alpha, \beta \rangle_{\mathcal{H}^*} = G(\alpha, \beta) + i \cdot \Omega(\alpha, \beta)$$

on the dual real Hilbert space  $\mathcal{H}_{\mathbf{R}}^*$  equipped with the dual complex structure  $J^*$ . Using the identification of  $\mathcal{H}_{\mathbf{R}}^*$  with  $\mathcal{H}_{\mathbf{R}}$  via the metric tensor  $g$ , the latter can be interpreted as a contravariant complex tensor on  $\mathcal{H}_{\mathbf{R}}$ . This tensor induces two real brackets of smooth functions on  $\mathcal{H}_{\mathbf{R}}$ :  $\{f, h\}_g = G(df, dh)$  and  $\{f, h\}_\omega = \Omega(df, dh)$ . The first one is the ‘Riemann–Jordan’ bracket associated with the contravariant version of the metric tensor  $g$  and the second is just the symplectic Poisson bracket associated with  $\omega$ . Of course both brackets can be extended to complex functions by complex linearity and give rise to the ‘total’ bracket

$$\{f, h\}_{\mathcal{H}} = \langle df, dh \rangle_{\mathcal{H}^*} = \{f, h\}_g + i \cdot \{f, h\}_\omega. \quad (1)$$

Fixing an orthonormal basis  $(e_k)$  of  $\mathcal{H}$  allows us to identify the Hermitian product  $\langle x, y \rangle_{\mathcal{H}}$  on  $\mathcal{H}$  with the canonical Hermitian product on  $\mathbf{C}^n$

$$\langle a, b \rangle_{\mathbf{C}^n} = \overline{a_k} b_k \quad (2)$$

(we use the convention of summation on repeated indices), the group  $U(\mathcal{H})$  of unitary transformations of  $\mathcal{H}$  with  $U(n)$ , its Lie algebra  $u(\mathcal{H})$  with  $u(n)$ , etc. In this picture  $(a_{jk})^\dagger = (\overline{a_{kj}})$  and  $(T^\dagger T)_{jk} = \langle \alpha_j, \alpha_k \rangle$ , where  $\alpha_k = (t_{jk}) \in \mathbf{C}^n$  are columns of the matrix  $T = (t_{jk})$ . The choice of the basis induces (global) coordinates  $(q_k, p_k)$ ,  $k = 1, \dots, n$ , on  $\mathcal{H}_{\mathbf{R}}$  by

$$\langle e_k, x \rangle_{\mathcal{H}} = (q_k + i \cdot p_k)(x)$$

in which  $\partial_{q_k}$  is represented by  $e_k$  and  $\partial_{p_k}$  by  $i \cdot e_k$ . Hence the complex structure reads

$$J = \partial_{p_k} \otimes dq_k - \partial_{q_k} \otimes dp_k,$$

the Riemannian tensor

$$g = (dq_k \otimes dq_k + dp_k \otimes dp_k) = \frac{1}{2}(dq_k \vee dq_k + dp_k \vee dp_k)$$

and the symplectic form

$$\omega = dq_k \wedge dp_k,$$

where  $x \vee y = x \otimes y + y \otimes x$  is the symmetric, and  $x \wedge y = x \otimes y - y \otimes x$  is the wedge product. In complex coordinates  $z_k = q_k + i \cdot p_k$  one can write the Hermitian product as the complex tensor

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} = d\bar{z}_k \otimes dz_k.$$

The contravariant tensor  $G + i \cdot \Omega$  has the form

$$G + i \cdot \Omega = (\partial_{q_k} \otimes \partial_{q_k} + \partial_{p_k} \otimes \partial_{p_k}) + i \cdot (\partial_{q_k} \otimes \partial_{p_k} - \partial_{p_k} \otimes \partial_{q_k})$$

or, in complex coordinates,

$$G + i \cdot \Omega = (\partial_{q_k} - i \cdot \partial_{p_k}) \otimes (\partial_{q_k} + i \cdot \partial_{p_k}) = 4\partial_{z_k} \otimes \partial_{\bar{z}_k}.$$

In other words,

$$\{f, h\}_g = \frac{\partial f}{\partial q_k} \frac{\partial h}{\partial q_k} + \frac{\partial f}{\partial p_k} \frac{\partial h}{\partial p_k}, \quad \{f, h\}_\omega = \frac{\partial f}{\partial q_k} \frac{\partial h}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial h}{\partial q_k},$$

and

$$\{f, h\}_{\mathcal{H}} = 4 \frac{\partial f}{\partial z_k} \frac{\partial h}{\partial \bar{z}_k}.$$

Every complex linear operator  $A \in gl(\mathcal{H})$  on  $\mathcal{H}$  induces the quadratic function

$$f_A(x) = \frac{1}{2} \langle x, Ax \rangle_{\mathcal{H}}.$$

The function  $f_A$  is real if and only if  $A$  is Hermitian,  $A = A^\dagger$ .

One important convention we want to introduce is that we will identify the space of Hermitian operators  $A = A^\dagger$  with the dual  $u^*(\mathcal{H})$  of the (real) Lie algebra  $u(\mathcal{H})$ , according to the pairing between Hermitian  $A \in u^*(\mathcal{H})$  and anti-Hermitian  $T \in u(\mathcal{H})$  operators

$$\langle A, T \rangle = \frac{i}{2} \cdot \text{Tr}(AT).$$

The multiplication by  $i$  establishes further a vector space isomorphism  $u(\mathcal{H}) \ni T \mapsto iT \in u^*(\mathcal{H})$  which identifies the adjoint and the coadjoint action of the group  $U(\mathcal{H})$ ,  $\text{Ad}_U(T) = UTU^\dagger$ . Under this isomorphism  $u^*(\mathcal{H})$  becomes a Lie algebra with the Lie bracket  $[A, B] = \frac{1}{i}[A, B]_-$ , where  $[A, B]_- = AB - BA$  is the commutator bracket, equipped additionally with the scalar product

$$\langle A, B \rangle_{u^*} = \frac{1}{2} \text{Tr}(AB) \quad (3)$$

and an additional algebraic operation, the Jordan product  $[A, B]_+ = AB + BA$ . The scalar product is invariant with respect to both the Lie bracket and the Jordan product (or bracket)

$$\langle [A, \xi], B \rangle_{u^*(\mathcal{H})} = \langle A, [\xi, B] \rangle_{u^*(\mathcal{H})}, \quad (4)$$

$$\langle [A, \xi]_+, B \rangle_{u^*(\mathcal{H})} = \langle A, [\xi, B]_+ \rangle_{u^*(\mathcal{H})}, \quad (5)$$

and it identifies once more  $u^*(\mathcal{H})$  with its dual,

$$u^*(\mathcal{H}) \ni A \mapsto \widehat{A} = \frac{1}{i} A \in u(\mathcal{H}),$$

so vectors with covectors. Under this identification the metric (3) corresponds to the invariant metric

$$\langle \widehat{A}, \widehat{B} \rangle_u = \frac{1}{2} \text{Tr}(AB) \quad (6)$$

on  $u(\mathcal{H})$  which can be viewed also as a contravariant metric on  $u^*(\mathcal{H})$ .

For a (real) smooth function  $f$  on  $\mathcal{H}_{\mathbf{R}}$  let us denote by  $\text{grad}_f$  and  $\text{Ham}_f$  the gradient and the Hamiltonian vector field associated with  $f$  and the Riemannian and symplectic tensor,

respectively. In other words,  $g(\cdot, \text{grad}_f) = df$  and  $\omega(\cdot, \text{Ham}_f) = df$  or  $\text{grad}_f = G(df, \cdot)$  and  $\text{Ham}_f = \Omega(df, \cdot)$ . Note that any  $A \in \mathfrak{gl}(\mathcal{H})$  induces a linear vector field  $\tilde{A}$  on  $\mathcal{H}$  by  $\tilde{A}(x) = Ax$ .

**Lemma 1.** For Hermitian  $A$  we have

$$\text{grad}_{f_A} = \tilde{A} \quad \text{and} \quad \text{Ham}_{f_A} = i\tilde{A}.$$

**Proof.** If  $\langle \cdot, \cdot \rangle$  denotes the pairing between vectors and covectors then

$$\begin{aligned} \langle df_A(x), y \rangle &= \frac{1}{2}(\langle y, Ax \rangle_{\mathcal{H}} + \langle x, Ay \rangle_{\mathcal{H}}) = \Re(\langle y, Ax \rangle_{\mathcal{H}}) \\ &= g(Ax, Ax) = \omega(y, iAx). \end{aligned}$$

□

**Corollary 1.** For all  $A, B \in \mathfrak{gl}(\mathcal{H})$  we have

$$\{f_A, f_B\}_{\mathcal{H}} = f_{2AB}. \quad (7)$$

In particular,

$$\{f_A, f_B\}_g = f_{AB+BA}, \quad (8)$$

$$\{f_A, f_B\}_{\omega} = f_{-i(AB-BA)}. \quad (9)$$

**Proof.** For Hermitian  $A, B$  we have

$$\begin{aligned} \{f_A, f_B\}_{\mathcal{H}}(x) &= g(\text{grad}_A(x), \text{grad}_B(x)) + i \cdot \omega(\text{Ham}_f(x), \text{Ham}_f(x)) \\ &= g(Ax, Bx) + i \cdot \omega(iAx, iBx) = \langle Ax, Bx \rangle_{\mathcal{H}} = \langle x, ABx \rangle_{\mathcal{H}} = 2f_{AB}(x). \end{aligned}$$

But  $2AB = (AB + BA) + i(-i(AB - BA))$ , where  $AB + BA = [A, B]_+$  and  $-i(AB - BA) = -i[A, B]_-$  are Hermitian; thus  $f_{[A, B]_+}$  and  $f_{-i[A, B]_-}$  are real, so the thesis holds for the Hermitian  $A, B$ . For general  $A, B$  it follows by complex linearity. □

The unitary action of  $U(\mathcal{H})$  on  $\mathcal{H}$  is in particular Hamiltonian and induces a momentum map  $\mu : \mathcal{H}_{\mathbb{R}} \rightarrow u^*(\mathcal{H})$ . The fundamental vector field associated with  $\frac{1}{i}A \in u(\mathcal{H})$ , where  $A \in u^*(\mathcal{H})$  is Hermitian, reads  $i\tilde{A}$ , since

$$\left. \frac{d}{dt} \right|_{t=0} \exp\left(-\frac{t}{i}A\right)(x) = iAx.$$

The Hamiltonian of  $i\tilde{A}$  is  $f_A$ , so the momentum map is defined by

$$\left\langle \mu(x), \frac{1}{i}A \right\rangle = f_A(x) = \frac{1}{2}\langle x, Ax \rangle_{\mathcal{H}}.$$

But by our convention

$$\left\langle \mu(x), \frac{1}{i}A \right\rangle = \frac{i}{2} \text{Tr} \left( \mu(x) \frac{1}{i}A \right) = \frac{1}{2} \text{Tr}(\mu(x)A),$$

so that  $\text{Tr}(\mu(x)A) = \langle x, Ax \rangle_{\mathcal{H}}$  and finally, in the Dirac notation,

$$\mu(x) = |x\rangle\langle x|. \quad (10)$$

Note that for  $A$  being Hermitian  $f_A$  is the pullback  $f_A = \mu^*(\widehat{A}) = \widehat{A} \circ \mu$ , where  $\widehat{A} = \langle A, \cdot \rangle_{u^*} = \frac{1}{i}A \in u(\mathcal{H})$ . The linear functions  $\widehat{A}$  generate  $T^*u^*(\mathcal{H})$ , so that (8) and (9) mean that the momentum map  $\mu$  relates contravariant tensors  $G$  and  $\Omega$  on  $\mathcal{H}$ , respectively, with the linear contravariant tensors  $R$  and  $\Lambda$  on  $u^*(\mathcal{H})$  corresponding to the Jordan and Lie bracket, respectively. The Riemann–Jordan tensor  $R$ , defined in the obvious way,

$$R(\xi)(\widehat{A}, \widehat{B}) = \langle \xi, [A, B]_+ \rangle_{u^*} = \frac{1}{2} \text{Tr}(\xi(AB + BA)), \quad (11)$$



is symmetric and the tensor,

$$\Lambda(\xi)(\widehat{A}, \widehat{B}) = \langle \xi, [A, B] \rangle_{u^*} = \frac{1}{2i} \text{Tr}(\xi(AB - BA)), \quad (12)$$

is the canonical Kostant–Kirillov–Souriau Poisson tensor on  $u^*(\mathcal{H})$ . They form together the complex tensor

$$(R + i \cdot \Lambda)(\xi)(\widehat{A}, \widehat{B}) = 2\langle \xi, AB \rangle_{u^*} = \text{Tr}(\xi AB) \quad (13)$$

and the momentum map relates this tensor with the dual Hermitian product:

$$\mu_*(G + i \cdot \Omega) = R + i \cdot \Lambda. \quad (14)$$

**Example.** For  $\mathcal{H} = \mathbb{C}^2$  consider an orthonormal basis in  $u^*(2)$  consisting of

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and the associated coordinates  $u, x, y, z$ , where  $u(A) = \frac{1}{2} \text{Tr}(UA)$ , etc. In these coordinates the Poisson tensor reads

$$\Lambda = 2(z\partial_x \wedge \partial_y + x\partial_y \wedge \partial_z + y\partial_z \wedge \partial_x)$$

and the Riemann–Jordan tensor reads

$$R = \partial_u \vee (2x\partial_x + 2y\partial_y + 2z\partial_z) + u(\partial_u \vee \partial_u + \partial_x \vee \partial_x + \partial_y \vee \partial_y + \partial_z \vee \partial_z).$$

The rank of  $\Lambda(u, x, y, z)$  is 0 if  $x^2 + y^2 + z^2 = 0$  and 2 if  $x^2 + y^2 + z^2 > 0$ . The rank of  $R(u, x, y, z)$  is 0 at  $(u, x, y, z) = 0$ ; it is 2 for  $u = 0$  and  $x^2 + y^2 + z^2 > 0$ ; it is 3 for  $x^2 + y^2 + z^2 = u^2 > 0$  and it is 4 for  $x^2 + y^2 + z^2 \neq u^2 > 0$ .

The image  $\mu(\mathcal{H} \setminus \{0\})$  is the cone

$$\mathcal{P}^1(\mathcal{H}) = \{|x\rangle\langle x| : x \neq 0\}$$

of non-negatively defined Hermitian operators  $\xi = |x\rangle\langle x|$  of rank 1. The operator  $\xi$  is proportional to the one-dimensional projection  $\xi/\|\xi\|$ , so  $\xi^2 = \|\xi\|\xi$ , where  $\|\xi\| = \|x\|^2$  is the operator norm of  $\xi$ . The manifold  $\mathcal{P}^1(\mathcal{H})$  is foliated by  $U(\mathcal{H})$ -coadjoint orbits being complex projective spaces  $\mathcal{D}_r^1(\mathcal{H}) = \{|x\rangle\langle x| : \|x\| = r\}$ ,  $r > 0$ . In particular, the momentum map image of the  $(2n-1)$ -dimensional sphere  $\mathcal{S}_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\|^2 = \langle x, x \rangle_{\mathcal{H}} = 1\}$  is the complex projective space  $\mathcal{D}^1(\mathcal{H}) = \{|x\rangle\langle x| : \|x\| = 1\}$  of one-dimensional projectors.

The coadjoint orbits  $\mathcal{O}$  in  $u^*(\mathcal{H})$  possess canonical symplectic forms  $\eta^{\mathcal{O}}$  which build together the Poisson structure  $\Lambda$ . These forms, as the inverses of  $\Lambda|_{\mathcal{O}}$ , are characterized by

$$\eta_{\xi}^{\mathcal{O}}([A, \xi], [B, \xi]) = \langle [A, \xi], B \rangle_{u^*(\mathcal{H})} = -\langle \xi, [A, B] \rangle_{u^*}. \quad (15)$$

Indeed, the vectors  $[A, \xi] = \frac{1}{i}[A, \xi]_-$  form the tangent space of the  $U(\mathcal{H})$ -orbit at  $\xi$  and  $\eta^{\mathcal{O}}$  is the inverse of  $\Lambda|_{\mathcal{O}}$ . Due to invariance of the scalar product on  $u^*(\mathcal{H})$ :

$$\Lambda_{\xi}(\widehat{A}, \widehat{B}) = \langle \xi, [A, B] \rangle_{u^*(\mathcal{H})} = \langle [\xi, A], B \rangle_{u^*(\mathcal{H})} = \langle [\xi, A], \widehat{B} \rangle. \quad (16)$$

Hence  $\#\Lambda_{\xi}(\widehat{A}) = [\xi, A]$  and

$$\#\eta_{\xi}^r([\xi, A]) = (\#\Lambda_{\xi})^{-1}([\xi, A]) = \widehat{A},$$

so

$$\begin{aligned} \eta_{\xi}^r([A, \xi], [B, \xi]) &= \langle \#\eta_{\xi}^r([A, \xi]), [B, \xi] \rangle = -\langle \widehat{A}, [B, \xi] \rangle \\ &= -\langle A, [B, \xi] \rangle_{u^*(\mathcal{H})} = -\langle \xi, [A, B] \rangle_{u^*(\mathcal{H})}. \end{aligned}$$

The image  $\#R(T^*u^*(\mathcal{H}))$  of the tensor  $R$  is not an involutive (generalized) distribution, so its inverse  $\sigma = R^{-1}$  can be understood only as a ‘partial’ covariant tensor on  $u^*(\mathcal{H})$ ,

i.e. as a ‘partial symmetric 2-form’ which at  $\xi \in u^*(\mathcal{H})$  is defined only on vectors from  $\#R_\xi(\mathbb{T}_\xi^*u^*(\mathcal{H}))$ . There is a completely analogous characterization of the tensor  $\sigma$  to that of  $\eta$ . Both characterizations can be summarized as follows.

**Proposition 1.**

(a) The symplectic form  $\eta^\mathcal{O}$  on the  $U(\mathcal{H})$ -orbit  $\mathcal{O}$  is characterized by

$$\eta_\xi^\mathcal{O}([A, \xi], [B, \xi]) = \langle [A, \xi], B \rangle_{u^*(\mathcal{H})} = -\langle \xi, [A, B] \rangle_{u^*}, \quad (17)$$

where  $A, B \in u^*(\mathcal{H})$  are arbitrary Hermitian operators.

(b) The ‘partial tensor’  $\sigma$  on  $u^*(\mathcal{H})$  is characterized by

$$\sigma_\xi([A, \xi]_+, [B, \xi]_+) = \langle [A, \xi]_+, B \rangle_{u^*(\mathcal{H})} = \langle \xi, [A, B]_+ \rangle_{u^*}, \quad (18)$$

where  $A, B \in u^*(\mathcal{H})$  are arbitrary Hermitian operators.

Let us observe that the ‘partial tensor’  $\sigma$ , when restricted to any  $\mathcal{D}_r^1(\mathcal{H})$ , induces a Riemannian structure  $\sigma^r$  which, together with the symplectic structure  $\eta^r = \eta^{\mathcal{D}_r^1(\mathcal{H})}$ , induces a Kähler structure.

**Proposition 2.**

(a) The tensor  $\sigma^r$  being the restriction of the ‘partial tensor’  $\sigma$  to the  $U(\mathcal{H})$ -orbit  $\mathcal{D}_r^1(\mathcal{H})$  through  $\xi = \mu(x)$ ,  $r^2 = \|\xi\|^2$ , is proportional to the original scalar product on  $u^*(\mathcal{H})$ :

$$\sigma_\xi^r([A, \xi], [B, \xi]) = \frac{1}{\|\xi\|^2} \langle [A, \xi], [B, \xi] \rangle_{u^*(\mathcal{H})}. \quad (19)$$

(b) The  $(1, 1)$ -tensor  $\mathcal{J}$  on  $\mathcal{P}^1(\mathcal{H})$ ,  $\mathcal{J}_\xi(A) = \frac{1}{\|\xi\|^2} [A, \xi]$ , satisfies  $\mathcal{J}^3 = -\mathcal{J}$  and induces a complex structure  ${}^r\mathcal{J}$  on every  $\mathcal{D}_r^1(\mathcal{H})$ . Moreover,

$$\eta_\xi^r([A, \xi], {}^r\mathcal{J}_\xi([B, \xi])) = \sigma_\xi^r([A, \xi], [B, \xi]), \quad (20)$$

and

$$\eta_\xi^r({}^r\mathcal{J}_\xi([A, \xi]), {}^r\mathcal{J}_\xi([B, \xi])) = \eta_\xi^r([A, \xi], [B, \xi]), \quad (21)$$

i.e.  $(\mathcal{D}_r^1(\mathcal{H}), {}^r\mathcal{J}, \sigma^r, \eta^r)$  is a Kähler manifold for each  $r > 0$ .

**Proof.** Observe first that, due to the Leibniz rule,

$$[A, \xi] = \frac{1}{\|\xi\|^2} [A, \xi^2] = \frac{1}{\|\xi\|^2} [[A, \xi], \xi]_+.$$

Then, in view of (18),

$$\sigma_\xi^r([A, \xi], [B, \xi]) = \frac{1}{\|\xi\|^2} \langle \xi, [[A, \xi], [B, \xi]]_+ \rangle_{u^*(\mathcal{H})} = \frac{1}{2\|\xi\|^2} \text{Tr}(\xi \circ [[A, \xi], [B, \xi]]_+).$$

But

$$\begin{aligned} \text{Tr}(\xi \circ [[A, \xi], [B, \xi]]_+) &= \text{Tr}(\xi \circ [A, \xi] \circ [B, \xi] + \xi \circ [B, \xi] \circ [A, \xi]) \\ &= \text{Tr}([A, \xi^2] \circ [B, \xi] - [A, \xi] \circ \xi \circ [B, \xi] + \xi \circ [B, \xi] \circ [A, \xi]) \\ &= \text{Tr}([A, \xi^2] \circ [B, \xi]) = \|\xi\| \text{Tr}([A, \xi] \circ [B, \xi]) \\ &= 2\|\xi\| \langle [A, \xi], [B, \xi] \rangle_{u^*(\mathcal{H})} \end{aligned}$$

that proves (19).

To prove that  $\mathcal{J}$  is a complex structure on every orbit, let us recall that  $\xi^2 = \|\xi\|\xi$ . Passing to  $\xi' = \xi/\|\xi\|$  if necessary, we can assume for all further calculations that  $\|\xi\| = 1$  so that  $\mathcal{J}_\xi(A) = [A, \xi]$ . Hence,

$$[[[A, \xi], \xi], \xi] = -\frac{1}{i}[(A\xi^2 - 2\xi A\xi + \xi^2 A), \xi] = -\frac{1}{i}[(A\xi^3 - \xi^3 A)] = -[A, \xi] \tag{22}$$

and (19) follows. Moreover, since vectors  $[A, \xi]$  form the tangent space  $T_\xi \mathcal{D}_r^1(\mathcal{H})$ , (22) shows that  $\mathcal{J}$  reduced to  $\mathcal{D}_r^1(\mathcal{H})$  is an almost-complex structure  ${}^r\mathcal{J}$ . We shall show that the Nijenhuis torsion of  ${}^r\mathcal{J}$  vanishes, so the structure is integrable. To do this, we must show that the distribution in the complexified tangent bundle  $T\mathcal{D}_r^1(\mathcal{H}) \otimes \mathbf{C}$  which corresponds to eigenvectors of complexified  ${}^r\mathcal{J}$  with the eigenvalue  $i$  is involutive. But this distribution is generated by complex vector fields  $\bar{T}$  for  $T \in gl(\mathcal{H})$ , where  $\bar{T}(\xi) = \xi T(1 - \xi)$ . Indeed,

$$\mathcal{J}_\xi(\xi T(1 - \xi)) = [\xi T(1 - \xi), \xi] = \frac{1}{i}(\xi T(1 - \xi)\xi - \xi^2 T(1 - \xi)) = i \cdot \xi T(1 - \xi)$$

and this is a generating set due to the decomposition

$$T = (\xi T\xi + (1 - \xi)T(1 - \xi)) + (1 - \xi)T\xi + \xi T(1 - \xi)$$

into eigenvectors of  $\mathcal{J}$  with eigenvalues  $0, -i$  and  $i$ , respectively. The bracket of vector fields  $[\bar{T}_1, \bar{T}_2]_{vf}$  reads

$$\begin{aligned} [\bar{T}_1, \bar{T}_2]_{vf}(\xi) &= \xi T_1(1 - \xi)T_2(1 - \xi) - \xi T_2\xi T_1(1 - \xi) - \xi T_2(1 - \xi)T_1(1 - \xi) \\ &\quad + \xi T_1\xi T_1(1 - \xi) = \xi(T_1T_2 - T_2T_1)(1 - \xi) = \overline{([T_1, T_2]_-)} \end{aligned}$$

that proves involutivity.

Finally, it is sufficient to combine (17) and (19) to get (20). Then

$$\begin{aligned} \eta_\xi^r({}^r\mathcal{J}_\xi([A, \xi]), {}^r\mathcal{J}_\xi([B, \xi])) &= \sigma_\xi^r({}^r\mathcal{J}_\xi([A, \xi]), [B, \xi]) = \langle {}^r\mathcal{J}_\xi([A, \xi]), [B, \xi] \rangle_{u^*(\mathcal{H})} \\ &= \langle [[A, \xi], \xi], [B, \xi] \rangle_{u^*(\mathcal{H})} = -\langle [[[A, \xi], \xi], \xi], B \rangle_{u^*(\mathcal{H})} \\ &= \langle [[A, \xi], B] \rangle_{u^*(\mathcal{H})} = \eta_\xi^r([A, \xi], [B, \xi]) \end{aligned}$$

that proves (21). □

**Proposition 3.** *There is an identification of the orthogonal complement of the vector  $x \in \mathcal{H}$  with the tangent space to the  $U(\mathcal{H})$ -orbit through  $\xi = \mu(x)$  in  $u^*(\mathcal{H})$ . For  $y, y' \in \mathcal{H}$  orthogonal to  $x$  with respect to the Hermitian product, the vectors  $(\mu_*)_x(y), (\mu_*)_x(y')$  are tangent to the orbit through  $\xi$  and*

$$\sigma_\xi^r((\mu_*)_x(y), (\mu_*)_x(y')) = g(y, y'), \tag{23}$$

$$\eta_\xi^r((\mu_*)_x(y), (\mu_*)_x(y')) = \omega(y, y'), \tag{24}$$

$${}^r\mathcal{J}_\mu(x)((\mu_*)_x(y)) = (\mu_*)_x(Jy). \tag{25}$$

**Proof.** Since

$$(\mu_*)_x(y) = P_y^x = |y\rangle\langle x| + |x\rangle\langle y|$$

can be written as  $P_y^x = [A_y, \xi]$ , where  $A_y$  is a Hermitian operator such that  $Ax = iy$  and  $Ay = -i\frac{\|y\|^2}{\|x\|^2}x$ , the operators  $P_y^x, P_{y'}^x$ , viewed as vectors in  $u^*(\mathcal{H})$ , are tangent to the orbit through  $\xi$ . Then, due to (27),

$$\begin{aligned} \sigma_\xi^r(P_y^x, P_{y'}^x) &= \frac{1}{2\|x\|^2} \text{Tr}(P_y^x \circ P_{y'}^x) = \frac{1}{2\|x\|^2} \text{Tr}(\|x\|^2 \cdot |y\rangle\langle y'| + \langle y, y' \rangle_{\mathcal{H}} \cdot |x\rangle\langle x|) \\ &= \frac{1}{2\|x\|^2} (\|x\|^2(\langle y', y \rangle_{\mathcal{H}} + \langle y, y' \rangle_{\mathcal{H}})) = \Re(\langle y, y' \rangle_{\mathcal{H}}) = g(y, y'). \end{aligned}$$

To prove (24), we use (17):

$$\begin{aligned} \eta_{\xi}^r(P_y^x, P_{y'}^x) &= -\langle \xi, [A_y, A_{y'}] \rangle_{u^*(\mathcal{H})} = -\frac{1}{2} \text{Tr}(\xi \circ [A_y, A_{y'}]) \\ &= -\frac{1}{2} \langle x, [A_y, A_{y'}]x \rangle_{\mathcal{H}} = -\frac{1}{2i} (\langle A_y x, A_{y'} x \rangle_{\mathcal{H}} - \langle A_{y'} x, A_y x \rangle_{\mathcal{H}}) \\ &= -\Im(\langle iy, iy' \rangle_{\mathcal{H}}) = \omega(y, y'). \end{aligned}$$

Finally, (25) follows directly from  $\omega(y', Jy) = g(y', y)$  and (20). □

The above theorem says that the Kähler manifold  $(\mathcal{D}_r^1(\mathcal{H}), {}^r\mathcal{J}, \sigma^r, \eta^r)$  comes from a sort of a ‘Kähler reduction’ of the original linear Kähler manifold  $(\mathcal{H}_{\mathbf{R}}, J, g, \omega)$ . In particular, the symplectic manifold  $\mathcal{D}_r^1(\mathcal{H})$  is the symplectic reduction of  $(\mathcal{H}_{\mathbf{R}}, \omega)$  with respect to the isotropic submanifold  $S_r = \{x \in \mathcal{H} : \|x\| = r\}$ . The characteristic foliation of  $\omega|_{S_r}$  consists of orbits of the group  $S^1 = \{z \in \mathbf{C} : |z| = 1\}$  acting on  $\mathcal{H}$  by multiplication. The fundamental vector field of this action is

$$-\tilde{I} = p_k \partial_{q_k} - q_k \partial_{p_k}$$

which is simultaneously a Killing vector field for the Riemannian metric  $g$ . Therefore  $g$  induces a Riemannian metric on  $\mathcal{D}_r^1(\mathcal{H})$ , etc.

### 3. Smooth manifold structure on $\mathcal{P}^k(\mathcal{H})$

Recall that the space of non-negatively defined operators from  $gl(\mathcal{H})$ , i.e. of those  $\rho \in gl(\mathcal{H})$  which can be written in the form  $\rho = T^\dagger T$  for a certain  $T \in gl(\mathcal{H})$ , we denote by  $\mathcal{P}(\mathcal{H})$ . It is a cone as being invariant with respect to the homoteties by  $\lambda$  with  $\lambda \geq 0$ . The set of density states  $\mathcal{D}(\mathcal{H})$  is distinguished in the cone  $\mathcal{P}(\mathcal{H})$  by the equation  $\text{Tr}(\rho) = 1$ , so we will regard  $\mathcal{P}(\mathcal{H})$  and  $\mathcal{D}(\mathcal{H})$  as embedded in  $u^*(\mathcal{H})$ .

The space  $\mathcal{D}(\mathcal{H})$  is a convex set in the affine hyperplane in  $u^*(\mathcal{H})$ , determined by the equation  $\text{Tr}(\tau) = 1$ . The tangent spaces to this affine hyperplane are therefore canonically identified with the space of Hermitian operators with trace 0. It is known that the set of extreme points of  $\mathcal{D}(\mathcal{H})$  coincides with the set  $\mathcal{D}^1(\mathcal{H})$  of pure states; i.e. the set of one-dimensional orthogonal projectors  $|x\rangle\langle x|$  (see corollary 3). Hence every element of  $\mathcal{D}(\mathcal{H})$  is a convex combination of points from  $\mathcal{D}^1(\mathcal{H})$ . The space  $\mathcal{D}^1(\mathcal{H})$  of pure states can be identified with the complex projective space  $\mathcal{P}\mathcal{H} \simeq \mathbf{C}P^{n-1}$  via the projection  $\mathcal{H} \setminus \{0\} \ni x \mapsto |x\rangle\langle x| \in \mathcal{D}^1(\mathcal{H})$  which identifies the points of the orbits of the  $\mathbf{C} \setminus \{0\}$ -group action by complex homoteties. We have already seen that  $\mathcal{D}^1(\mathcal{H})$  is canonically a Kähler manifold. This will be the starting point for the study of geometry of the set  $\mathcal{D}(\mathcal{H})$  of all density states.

The (co)adjoint action of the group  $U(\mathcal{H})$  in  $u^*(\mathcal{H})$  induces its action on the positive cone  $\mathcal{P}(\mathcal{H})$  and on the space of density states. This action is transitive on pure states but it is no longer transitive on subsets  $\mathcal{D}^k(\mathcal{H})$ ,  $k > 1$ , where  $\mathcal{D}^k(\mathcal{H}) = \mathcal{D}(\mathcal{H}) \cap \mathcal{P}^k(\mathcal{H})$  and  $\mathcal{P}^k(\mathcal{H})$  consists of non-negative operators of rank  $k$ . The rank is understood clearly as the rank of the corresponding operator (or matrix, if a basis in  $\mathcal{H}$  is chosen). The intersection of  $\mathcal{D}(\mathcal{H})$  with any Weyl chamber in a Cartan subalgebra in  $u^*(\mathcal{H})$  is an  $(n - 1)$ -dimensional simplex, while the intersection of  $\mathcal{D}^k(\mathcal{H})$  is the  $(k - 1)$ -skeleton of this simplex. However, the dimension of the orbit may vary even for points from a chosen  $\mathcal{D}^k(\mathcal{H})$  if  $k > 1$ . Thus, the set of density states is a union of smooth manifolds—orbits of  $U(\mathcal{H})$ —but the differentiable structure of the stratum  $\mathcal{D}^k(\mathcal{H})$  is *a priori* not clear (for  $k > 1$ ), since the decomposition into orbits is not a regular foliation; i.e.  $\mathcal{D}^k(\mathcal{H})$  is the union of a family of various submanifolds of  $u^*(\mathcal{H})$  which differ even by dimensions. By the differential structure we mean here the differential structure

inherited from  $u^*(\mathcal{H})$ , so that the smooth curves in  $\mathcal{D}(\mathcal{H})$  and hence the tangent spaces are uniquely defined.

Our aim in this section is to understand this differential structure. Of course, the interior of  $\mathcal{D}(\mathcal{H})$ , namely  $\mathcal{D}^n(\mathcal{H})$ , is an open subset, so a submanifold in the affine subspace of trace = 1 Hermitian operators and the real question is only the boundary, consisting of those density states  $\rho$  for which  $\det(\rho) = 0$ . The best situation would be if the boundary were a submanifold, but this is not true in dimensions  $n > 2$  as will be shown later. The stratification into  $U(\mathcal{H})$ -orbits is too small, since, as it will appear later, the subsets  $\mathcal{D}^k(\mathcal{H})$  are coarser submanifolds in  $u^*(\mathcal{H})$ . We will show also that the stratification by rank is the maximal one in the sense that the vectors tangent to  $\mathcal{D}(\mathcal{H})$  at  $\rho \in \mathcal{D}^k(\mathcal{H})$  must be tangent to  $\mathcal{D}^k(\mathcal{H})$  itself, so the largest  $u^*(\mathcal{H})$ -submanifold through  $\rho \in \mathcal{D}^k(\mathcal{H})$  contained in  $\mathcal{D}(\mathcal{H})$  is  $\mathcal{D}^k(\mathcal{H})$ .

We start by fixing an orthonormal basis in  $\mathcal{H}$  which allows us to identify  $u^*(\mathcal{H})$  with the space  $u^*(n)$  of Hermitian  $(n \times n)$ -matrices which is canonically an  $n^2$ -dimensional real manifold with respect to the identification

$$u^*(n) \ni (a_{ij}) \mapsto ((a_{ii})_1^n, (a_{ij})_{i < j}) \in \mathbf{R}^n \times \mathbf{C}^{n(n-1)/2}.$$

By  $\mathcal{P}(n)$  we denote the space of non-negatively defined matrices from  $u^*(n)$  by  $\mathcal{P}^k(n)$  the subset of rank  $k$  matrices from  $\mathcal{P}(n)$ , etc. Let us denote by  $\mathcal{P}_J^k(n)$  the set of matrices  $A = (a_{ij})_{i,j=1}^n \in \mathcal{P}(n)$  being of rank  $k$  and such that the minor  $\det[(a_{rs})_{r,s \in J}]$  associated with a set of indices  $J = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$  is non-vanishing<sup>4</sup>. The next lemma shows that any matrix from  $\mathcal{P}_J^k(n)$  can be reconstructed from its rows (or columns, since it is Hermitian) indexed by  $J$ .

**Lemma 2.** *Let  $A = (a_{ij})_{i,j=1}^n \in \mathcal{P}_J^k(n)$ , so that the matrix  $(a_{rs})_{r,s \in J}$  has the inverse  $(a^{rs})_{r,s \in J}$ . Then the matrix  $A$  is uniquely determined by  $\{(a_{ij}) : i \in J, j = 1, \dots, n\}$  according to the formula*

$$a_{ij} = \sum_{r,s \in J} a_{ir} a^{rs} \overline{a_{js}}. \quad (26)$$

**Proof.** The matrix  $A$  being non-negatively defined is of the form  $T^\dagger T$  for certain  $(n \times n)$ -matrix  $T$ , so that  $a_{ij}$  is the Hermitian product  $\langle \alpha_i, \alpha_j \rangle$  of columns of  $T$  with respect to the standard Hermitian product (2). The matrix  $T$  is not uniquely determined. However, the fact that  $A$  is of rank  $k$  with the non-vanishing minor associated with  $J$  means that the columns  $\alpha_j$ ,  $j \in J$  are linearly independent and span the rest of the columns of  $T$ . But the Hermitian product on the subspace in  $\mathbf{C}^n$  spanned by  $\{\alpha_j : j \in J\}$  is given by the formula

$$\langle x, y \rangle_{\mathbf{C}^n} = \sum_{r,s \in J} \langle x, \alpha_r \rangle_{\mathbf{C}^n} \alpha^{rs} \langle \alpha_s, y \rangle_{\mathbf{C}^n}, \quad (27)$$

where  $(\alpha^{rs})_{r,s \in J}$  is the inverse of the matrix  $(\langle \alpha_r, \alpha_s \rangle_{\mathbf{C}^n})_{r,s \in J}$ . The proof of (27) is immediate, since the rhs of (27) is  $\mathbf{C}$ -linear with respect to  $y$ , anti-linear with respect to  $x$  and equals  $\langle \alpha_i, \alpha_j \rangle_{\mathbf{C}^n}$  for  $x = \alpha_i$ ,  $y = \alpha_j$ ,  $i, j \in J$ , by definition. Since  $a_{ij} = \langle \alpha_i, \alpha_j \rangle_{\mathbf{C}^n}$ , we get the formula (26) directly from (27).  $\square$

**Remark.** It is worth noticing that the formula (27) is similar to the one describing the Dirac bracket on constraint manifolds induced by second class constraints.

For  $J$  as above define a linear map

$$\Phi_J : u^*(n) \rightarrow u^*(k) \times \mathbf{C}^{(n-k)k} \simeq \mathbf{R}^k \times \mathbf{C}^{(2nk-k^2-k)/2} \simeq \mathbf{R}^{2nk-k^2}$$

<sup>4</sup> The set  $J$  is not to be confused in the following with the complex structure denoted accidentally by the same letter. From the context, however, the notion of  $J$  is always obvious.

by

$$\Phi_J((a_{ij})_{i,j=1}^n) = ((a_{ij})_{i,j \in J}, (a_{rs})_{r \notin J, s \in J}). \tag{28}$$

In particular, if we work with the principal minor, i.e.  $J = \{1, \dots, k\}$ , then  $\Phi_J$  associates with a Hermitian matrix its first  $k$  columns with removed, say, upper-triangular part which is irrelevant due to hermicity or, equivalently, its first  $k$  rows with the removed lower-triangular part.

For  $A \in u^*(n)$  by  $\Phi_{J,A}$  we denote the map  $\Phi_{J,A}(X) = \Phi_J(X) - \Phi_J(A)$ :

$$\Phi_{J,A}((x_{ij})_{i,j=1}^n) = ((x_{ij} - a_{ij})_{i,j \in J}, (x_{rs} - a_{rs})_{r \notin J, s \in J}). \tag{29}$$

With some abuse of notation, its restriction to  $\mathcal{P}^k(n)$  we will denote by the same symbol. It is clear from the above lemma that the map  $\Phi_J$  is continuous and injective on  $\mathcal{P}_J^k(n)$ . Thus, for  $A \in \mathcal{P}_J^k(n)$ , the map  $\Phi_{J,A}$  is also continuous and injective on  $\mathcal{P}_J^k(n)$ .

Conversely, every point

$$((y_{ij})_{i,j \in J}, (y_{rs})_{r \notin J, s \in J})$$

of  $u^*(k) \times \mathbf{C}^{(n-k)k} \simeq \mathbf{R}^{2nk-k^2}$ , sufficiently close to 0, is the value  $\Phi_{J,A}(X)$  for a certain  $X \in \mathcal{P}_J^k(n)$ . Indeed, adding a small Hermitian matrix to  $(a_{ij})_{i,j \in J}$  will not change its invertibility. Hence we have to reconstruct  $X$  out of  $\Phi_J(X)$ , i.e out of the columns (and rows, since  $X$  should be Hermitian) with indices belonging to  $J$  and knowing that  $(x_{ij})_{i,j \in J}$  has an inverse, say,  $(x^{rs})_{r,s \in J}$ . Here  $x_{ij} = a_{ij} + y_{ij}$  for  $j \in J$ . An obvious choice is the formula (26), i.e.

$$x_{ij} = \sum_{r,s \in J} x_{ir} x^{rs} \overline{x_{js}}.$$

The only thing to be checked is that  $X = (x_{ij})_{i,j=1}^n$  defined in this way is non-negatively defined and of rank  $k$ . Assume, for simplicity of notation, that  $J = \{1, \dots, k\}$ . First, we can find vectors  $\beta_1, \dots, \beta_k \in \mathbf{C}^k$  such that

$$x_{ij} = \langle \beta_i, \beta_j \rangle_{\mathbf{C}^k} \tag{30}$$

for  $i, j = 1, \dots, k$ . This can be done up to a unitary transformation. For example,  $\beta_i$  can be columns of the matrix  $\sqrt{(x_{ij})_{i,j=1}^k}$ . Then, we find (this time unique) vectors  $\beta_{k+1}, \dots, \beta_n \in \mathbf{C}^k$  satisfying the conditions  $x_{ij} = \langle \beta_i, \beta_j \rangle_{\mathbf{C}^k}$ ,  $i = k+1, \dots, n$ ,  $j = 1, \dots, k$ . It is easy to see now that, due to the formula (27), we have (30) for all  $i, j = 1, \dots, n$ . This immediately implies that  $X$  is non-negatively defined and of rank  $k$ . Moreover, since

$$x_{ij} = \sum_{r,s \in J} (a_{ir} + y_{ir}) a_y^{rs} (\overline{y_{js}} + \overline{a_{js}}), \tag{31}$$

where  $(a_y^{rs})_{r,s \in J}$  is the inverse of the matrix  $(a_{rs} + y_{rs})_{r,s \in J}$ , the matrix elements  $x_{ij}$  rationally depend on  $y_{ml}$ , so that  $\Phi_{J,A}^{-1}$  is smooth, thus also regular, as a function from a neighbourhood of 0 in  $\mathbf{R}^{2nk-k^2}$  into  $u^*(n)$ , so  $\mathcal{P}^k(n)$  is a submanifold in  $u^*(n)$ . To see the image of the differential of  $\Phi_{J,A}^{-1}$  at 0, i.e the tangent space  $T_A \mathcal{P}^k(n)$ , let us consider the linear (with respect to  $y$ ) part  $(v_{ij})$  of the rhs of (31):

$$v_{ij} = \sum_{r,s \in J} (y_{ir} a^{rs} \overline{a_{js}} - a_{ir} a^{rm} y_{ml} a^{ls} \overline{a_{js}} + a_{ir} a^{rs} \overline{y_{js}}). \tag{32}$$

To see this better, let us change the orthogonal basis of  $\mathbf{C}^n$  for such that  $J = \{1, \dots, k\}$  and  $A$  is diagonal,  $a_{ii} = \lambda_i$ ,  $\lambda_i = 0$  for  $i > k$ . Then one can easily find that (32) takes the form

$$v_{ij} = \begin{cases} 0, & \text{if } i, j > k \\ y_{ij}, & \text{if } j \leq k. \end{cases} \tag{33}$$

This means that in the image are arbitrary Hermitian matrices  $V = (v_{ij})_{i,j=1}^n$  such that  $v_{ij} = 0$  for  $i, j > k$  that can be written in a coordinate-free way as  $\langle Vx, y \rangle_{\mathbf{C}^n} = 0$  for all  $x, y \in \text{Ker}(A)$ . Note that the manifold  $\mathcal{P}^k(\mathcal{H})$  is connected. Indeed, it consists of connected orbits of the group  $U(\mathcal{H})$  which meet a Weyl chamber as the  $(k - 1)$ -dimensional skeleton of a simplex. However, the connected components of this skeleton are identified by the action of the Weyl group, so they form topologically a  $(k - 1)$ -dimensional simplex which is obviously connected. Therefore we have proved the following.

**Theorem 1.** *Let  $A \in \mathcal{P}_J^k(n)$ . Then the map  $\Phi_{J,A} : \mathcal{P}^k(n) \rightarrow \mathbf{R}^{2nk-k^2}$  defined by (29) is a local homeomorphism from a neighbourhood of  $A$  in  $\mathcal{P}^k(n)$  onto a neighbourhood of  $0$  in  $u^*(k) \times \mathbf{C}^{(n-k)k} \simeq \mathbf{R}^{2nk-k^2}$ . Moreover, the collection of the maps  $\Phi_{J,A}^{-1} : \mathcal{W}_{J,A} \rightarrow \mathcal{P}^k(n) \subset u^*(n)$  defined on sufficiently small neighbourhoods  $\mathcal{W}_{J,A}$  of  $0$  by the formula (31) constitutes a smooth manifold structure on  $\mathcal{P}^k(n)$  which makes it into a smooth and connected submanifold of  $u^*(n)$ . The tangent space  $T_A \mathcal{P}^k(n)$ , viewed as a subspace of  $u^*(n)$ , consists of matrices  $V \in u^*(n)$  satisfying  $\langle Vx, y \rangle_{\mathbf{C}^n} = 0$  for all  $x, y \in \text{Ker}(A)$ .*

**Remark.** In section 5 we obtain the manifold structure on  $\mathcal{P}^k(n)$  much simpler as the structure of an  $GL(n, \mathbf{C})$ -orbit. But we find that lemma 2 and theorem 1 are of some interest *per se* providing explicit coordinate systems.

The next theorem shows that smooth curves in  $u^*(n)$  which lay in  $\mathcal{P}(n)$  cannot cross  $\mathcal{P}^k(n)$  transversally, i.e.  $\mathcal{P}^k(n)$  is in a sense an edge for  $\mathcal{P}^{k+1}(n)$  if  $k < n - 1$ .

**Theorem 2.** *Let  $\gamma : \mathbf{R} \rightarrow u^*(n)$  be a smooth curve in the space of Hermitian matrices which lies entirely in  $\mathcal{P}(n)$ . Then  $\gamma$  is tangent to the stratum  $\mathcal{P}^k(n)$  it belongs, i.e.  $\gamma(t) \in \mathcal{P}^k(n)$  implies  $\dot{\gamma}(t) \in T_{\gamma(t)} \mathcal{P}^k(n)$ .*

**Proof.** Of course, it is enough to prove the above for an arbitrary  $t \in \mathbf{R}$ , say,  $t = 0$ . Assume therefore that  $A = \gamma(0) \in \mathcal{P}^k(n)$ . Take  $x \in \text{Ker}(A)$ . Since

$$\left\langle \frac{\gamma(\Delta t) - \gamma(0)}{\Delta t} x, x \right\rangle \geq 0$$

for  $\Delta t \geq 0$ , we have  $\langle \dot{\gamma}(0)x, x \rangle \geq 0$ . Taking in turn  $\Delta t \leq 0$  we see in a similar way that  $\langle \dot{\gamma}(0)x, x \rangle \leq 0$ , so

$$\langle \dot{\gamma}(0)x, x \rangle = 0. \quad (34)$$

By polarization of (34) we get

$$\langle \dot{\gamma}(0)x, y \rangle + \langle \dot{\gamma}(0)y, x \rangle = 0 \quad (35)$$

for all  $x, y \in \text{Ker}(A)$ . But  $\dot{\gamma}(0)$  is Hermitian, so

$$\langle \dot{\gamma}(0)y, x \rangle = \langle y, \dot{\gamma}(0)x \rangle$$

and (35) yields that the real part  $\Re(\langle \dot{\gamma}(0)x, y \rangle)$  is 0 for all  $x, y \in \text{Ker}(A)$ . On the other hand, the kernel of  $A$  is a complex subspace and

$$\Re(\langle \dot{\gamma}(0)x, i \cdot y \rangle) = \Im(\langle \dot{\gamma}(0)x, y \rangle)$$

so

$$\langle \dot{\gamma}(0)x, y \rangle = 0 \quad (36)$$

for all  $x, y \in \text{Ker}(A)$ . But, according to theorem 1, (36) means that  $\dot{\gamma}(0) \in T_A \mathcal{P}^k(n)$ .  $\square$

#### 4. Smooth stratification of density states

The set  $\mathcal{D}(\mathcal{H})$  of density states on  $\mathcal{H}$  is the intersection of the cone  $\mathcal{P}(\mathcal{H})$  with the affine subspace  $\{A \in u^*(\mathcal{H}) : \text{Tr}(A) = 1\}$  or, in other words, it is the level set of the function  $\text{Tr} : \mathcal{P}(\mathcal{H}) \rightarrow \mathbf{R}$  corresponding to the value 1. Since  $\text{Tr}(t\rho) = t \text{Tr}(\rho)$  and  $\mathcal{P}^k$  is invariant with respect to homoteties with positive  $t$ , it is clear that  $\text{Tr}$  is a regular function on each  $\mathcal{P}^k(\mathcal{H})$  so that  $\mathcal{D}^k(\mathcal{H})$  is canonically a smooth manifold. Since topologically  $\mathcal{P}^k(\mathcal{H}) \simeq \mathcal{D}^k(\mathcal{H}) \times \mathbf{R}$ , the manifolds  $\mathcal{D}^k(\mathcal{H})$  are connected. All these observations together with theorems 1 and 2 can be summarized in the following.

**Theorem 3.** *The spaces  $\mathcal{D}^k(\mathcal{H})$  of density states of rank  $k, k = 1, \dots, n$ , are smooth and connected submanifolds in  $u^*(\mathcal{H})$  of (real) dimension  $2nk - k^2 - 1$ . The tangent space  $T_\rho \mathcal{D}^k(\mathcal{H})$  is characterized as the space of those Hermitian operators  $T$  of trace 0 which satisfy  $\langle Tx, y \rangle = 0$  for all  $x, y \in \text{Ker}(\rho)$ . Moreover, the stratification into submanifolds  $\mathcal{D}^k(\mathcal{H})$  is maximal in the sense that every smooth curve in  $u^*(\mathcal{H})$ , which lies entirely in  $\mathcal{D}(\mathcal{H})$ , at every point is tangent to the strata  $\mathcal{D}^k(\mathcal{H})$  to which it actually belongs.*

**Corollary 2.** *The boundary  $\partial \mathcal{D}(\mathcal{H}) = \bigcup_{k < n} \mathcal{D}^k(\mathcal{H})$  of the set of density states is not a smooth submanifold of  $u^*(\mathcal{H})$  if  $n = \dim \mathcal{H} > 2$ .*

**Proof.** If  $n > 2$  then the boundary  $\partial \mathcal{D}(\mathcal{H})$  has at least two different strata and the vectors orthogonal to, say, the stratum  $\mathcal{D}^1(\mathcal{H})$  of pure states are not tangent to  $\partial \mathcal{D}(\mathcal{H})$ . But the dimension of  $\mathcal{D}^1(\mathcal{H})$  is smaller than the topological dimension of  $\partial \mathcal{D}(\mathcal{H})$ .  $\square$

**Remark.** It is well known that for  $n = 2$  the convex set of density states is affinely equivalent to the three-dimensional ball and its boundary—to the two-dimensional sphere, so that it is a smooth manifold.

In the last problem concerning the geometry of density states we will consider is the question of affine parts of the manifolds  $\mathcal{D}^k(\mathcal{H})$ . It is motivated by the fact that the set  $\mathcal{D}^1(\mathcal{H})$  of pure states is exactly the set of extremal elements of  $\mathcal{D}(\mathcal{H})$ , so it does not contain intervals, but the other strata  $\mathcal{D}^k(\mathcal{H})$  with  $k > 1$  must do as shows the following theorem. Recall that a non-empty closed convex subset  $K_0$  of a closed convex set  $K$  is called a *face* (or extremal subset) of  $K$  if any closed segment in  $K$  with an interior point in  $K_0$  lies entirely in  $K_0$ ; a point  $x$  is called an *extreme* point of  $K$  if the set  $\{x\}$  is a face of  $K$ .

**Theorem 4.** *If  $\rho \in \mathcal{D}^k(\mathcal{H})$  then the affine space in  $u^*(\mathcal{H})$  which is tangent to  $\mathcal{D}^k(\mathcal{H})$  at  $\rho$  intersects  $\mathcal{D}(\mathcal{H})$  along a  $(k^2 - 1)$ -dimensional convex body which is affinely equivalent to the set  $\mathcal{D}(k)$  of density states in dimension  $k$ . This convex body is exactly the face of  $\mathcal{D}(\mathcal{H})$  at  $\rho$ . In other words, the face of  $\mathcal{D}(\mathcal{H})$  at  $\rho \in \mathcal{D}^k(\mathcal{H})$  is affinely equivalent to  $\mathcal{D}(k)$ .*

**Proof.** Let us take coordinates in  $u^*(\mathcal{H})$ , i.e. let us choose an orthonormal basis in  $\mathcal{H}$  in which  $\rho$  is represented by a diagonal matrix  $(\rho_{ij})$ ,  $\rho_{ij} = \delta_j^i \lambda_i$ , where  $\lambda_i = 0$  for  $i > k$ . According to the form of  $T_\rho \mathcal{D}^k(\mathcal{H})$ , matrices  $(x_{ij})$  which belong to  $\rho + T_\rho \mathcal{D}^k(\mathcal{H})$  have entries  $x_{ij}$  with  $i, j > k$  equal to 0. If they belong as well to  $\mathcal{D}(\mathcal{H})$ , also  $x_{ij} = 0$  if  $i > k$  or  $j > k$ . Indeed, since  $x_{ij} = \langle \alpha_i, \alpha_j \rangle$  for certain vectors  $z_i \in \mathbf{C}^n$ , we have  $x_{ii} = \|\alpha_i\|^2 = 0$ , so  $\alpha_i = 0$ , for  $i > k$ , and further  $x_{ij} = \langle \alpha_i, \alpha_j \rangle = 0$  if  $i > k$  or  $j > k$ . In other words, the only non-zero part of  $X$  is the block  $(x_{ij})_{i,j=1}^k$  which is therefore an element of  $\mathcal{D}(k)$ . Conversely, every matrix  $X$  with such a block form belongs simultaneously to  $\mathcal{D}(\mathcal{H})$  and, since  $(X - \rho)_{ij} = 0$  for  $i, j > k$ , to  $\rho + T_\rho \mathcal{D}^k(\mathcal{H})$ . To see that  $(\rho + T_\rho \mathcal{D}^k(\mathcal{H})) \cap \mathcal{D}(\mathcal{H})$  is exactly the face of  $\mathcal{D}(\mathcal{H})$  at  $\rho$ , consider a segment in  $\mathcal{D}(\mathcal{H})$  for which  $\rho$  is an interior point. The open segment is clearly a smooth



curve in  $\mathcal{D}(\mathcal{H})$ , so, in view of theorem 3, it is tangent to  $\mathcal{D}^k(\mathcal{H})$  at  $\rho$ ; thus belongs entirely to  $\rho + T_\rho \mathcal{D}^k(\mathcal{H})$ .  $\square$

**Corollary 3.** *Extremal points of  $\mathcal{D}(\mathcal{H})$  are exactly pure states.*

**5. Geometry of  $u^*(\mathcal{H})$**

Let us mention that a major part of what has been said about the differential structure of the space  $\mathcal{P}^k(\mathcal{H})$  of rank- $k$  positive operators can be repeated for the space of all rank- $k$  Hermitian operators. Denote by  $u_{k_+,k_-}^*(\mathcal{H})$  the set of those Hermitian operators  $\xi$  whose spectrum contains  $k_+$  positive and  $k_-$  negative eigenvalues (counted with multiplicities), respectively. Thus the rank of  $\xi$  is  $k = k_+ + k_-$  and  $\mathcal{P}^k(n) = u_{k,0}^*(n)$ .

Fixing an orthogonal basis in  $\mathcal{H}$  will identify  $u_{k_+,k_-}^*(\mathcal{H})$  with the space  $u_{k_+,k_-}^*(n)$  of  $n \times n$  Hermitian matrices of rank  $k$  with the corresponding spectrum. Denote by  $D_{k_-}^{k_+}$  the diagonal matrix  $\text{diag}(1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$  with 1 coming  $k_+$ -times and  $-1$  coming  $k_-$ -times. Denote by  $\langle \cdot, \cdot \rangle_{k_+,k_-}$  the ‘semiHermitian’ product in  $\mathbf{C}^n$  represented by  $D_{k_-}^{k_+}$ :

$$\langle a, b \rangle_{k_+,k_-} = \sum_{j=1}^{k_+} \bar{a}_j b_j - \sum_{j=k_++1}^{k_++k_-} \bar{a}_j b_j. \tag{37}$$

It is easy to see the following.

**Proposition 4.** *Any  $\xi = (a_{ij}) \in u_{k_+,k_-}^*(n)$  can be written in the form  $\xi = T^\dagger D_{k_-}^{k_+} T$  for certain  $T \in GL(n, \mathbf{C})$ . In other words the entries of the matrix  $\xi$  are semiHermitian products  $a_{ij} = \langle \alpha_i, \alpha_j \rangle_{k_+,k_-}$ , where  $\alpha_i$  denotes the  $i$ th column of  $T$ .*

**Proof.** We can diagonalize  $\xi$  by means of an unitary matrix  $U$ ,

$$U \xi U^\dagger = \text{diag}(\lambda_1, \dots, \lambda_n),$$

where  $\lambda_1 \geq \dots \geq \lambda_n$ , so  $\lambda_1, \dots, \lambda_{k_+} > 0$  and  $\lambda_{k_++1}, \dots, \lambda_{k_++k_-} < 0$ . Hence  $\xi = T^\dagger D_{k_-}^{k_+} T$  for  $T = CU$  with

$$C = \text{diag}(\sqrt{|\lambda_1|}, \dots, \sqrt{|\lambda_{k_++k_-}|}, 1, \dots, 1). \tag{38}$$

Now, we can reformulate lemma 2 for  $u_{k_+,k_-}^*(n)$  instead of  $\mathcal{P}^k(n)$ . The proof is essentially the same with the difference that we use the semiHermitian product  $\langle \cdot, \cdot \rangle_{k_+,k_-}$  in  $\mathbf{C}^n$  instead of  $\langle \cdot, \cdot \rangle_{\mathbf{C}^n}$ .

**Lemma 3.** *Let  $\xi = (a_{ij})_{i,j=1}^n \in u_{k_+,k_-}^*(n)$ . Assume that the matrix  $(a_{rs})_{r,s \in J}$  has the inverse  $(a^{rs})_{r,s \in J}$  for certain  $k = (k_+ + k_-)$ -element subset  $J = \{j_1, \dots, j_k\} \subset \{1, \dots, n\}$ . Then the matrix  $\xi$  is uniquely determined by  $\{(a_{ij}) : i \in J, j = 1, \dots, n\}$  according to the formula*

$$a_{ij} = \sum_{r,s \in J} a_{ir} a^{rs} \bar{a}_{js}. \tag{38}$$

One can now prove that  $u_{k_+,k_-}^*(\mathcal{H})$  are submanifolds of  $u^*(\mathcal{H})$  in completely parallel way to the case of  $\mathcal{P}^k(\mathcal{H})$ . However, proposition 4 suggests an easier (although less constructive) way to do it. Namely, we can see  $u_{k_+,k_-}^*(\mathcal{H})$  as an orbit of a natural  $GL(\mathcal{H})$  action on  $u^*(\mathcal{H})$ .

**Theorem 5.** *The family*

$$\{u_{k_+,k_-}^*(\mathcal{H}) : k_+, k_- \geq 0, k = k_+ + k_- \leq n\} \tag{39}$$

of subsets of  $u^*(\mathcal{H})$  is exactly the family of orbits of the smooth action of the group  $GL(\mathcal{H})$  given by

$$GL(\mathcal{H}) \times u^*(\mathcal{H}) \ni (T, \xi) \mapsto T\xi T^\dagger \in u^*(\mathcal{H}). \tag{40}$$

In particular, every  $u_{k_+,k_-}^*(\mathcal{H})$  is a connected submanifold of  $u^*(\mathcal{H})$  and the tangent space to  $u_{k_+,k_-}^*(\mathcal{H})$  at  $\xi$  is characterized by

$$B \in T_\xi u_{k_+,k_-}^*(\mathcal{H}) \Leftrightarrow \forall x, y \in \text{Ker}(\xi) [\langle Bx, y \rangle_{\mathcal{H}} = 0]. \tag{41}$$

Moreover, the following are equivalent:

- (1)  $u_{k_+,k_-}^*(\mathcal{H})$  intersects  $\mathcal{P}(\mathcal{H})$ ,
- (2)  $u_{k_+,k_-}^*(\mathcal{H})$  is contained in  $\mathcal{P}(\mathcal{H})$ ,
- (3)  $k_- = 0$ ,
- (4)  $u_{k_+,k_-}^*(\mathcal{H}) = \mathcal{P}^k(\mathcal{H})$ ,  $k = k_+ + k_-$ .

**Proof.** The proof that (40) is a group of smooth action is straightforward. Proposition 4 shows that  $u_{k_+,k_-}^*(\mathcal{H})$  is contained in the  $GL(\mathcal{H})$ -orbit of  $D_{k_-}^{k_+}$ .

On the other hand, although the spectrum is not fixed on every  $GL(\mathcal{H})$ -orbit, the number of positive and negative eigenvalues (counted with multiplicities) are fixed along the orbit. Indeed, if  $\langle x, \xi x \rangle_{\mathcal{H}} > 0$  (respectively,  $\langle x, \xi x \rangle_{\mathcal{H}} < 0$ ) for  $x$  in a  $k_+$ -dimensional (respectively,  $k_-$ -dimensional) linear subspace  $V_+$  (respectively  $V_-$ ), then  $\langle x, T\xi T^\dagger x \rangle_{\mathcal{H}} = \langle T^\dagger x, \xi T^\dagger x \rangle_{\mathcal{H}} > 0$  (respectively  $\langle x, T\xi T^\dagger x \rangle_{\mathcal{H}} = \langle T^\dagger x, \xi T^\dagger x \rangle_{\mathcal{H}} < 0$ ) for  $x$  in a  $k_+$ -dimensional (respectively  $k_-$ -dimensional) linear subspace  $(T^\dagger)^{-1}(V_+)$  (respectively  $(T^\dagger)^{-1}(V_-)$ ).

The corresponding infinitesimal action of  $v \in gl(\mathcal{H})$  is  $\xi \mapsto v\xi + \xi v^\dagger$  and the operators  $\xi_v = v\xi + \xi v^\dagger$  clearly satisfy  $\langle x, \xi_v y \rangle_{\mathcal{H}} = 0$  for all  $x, y \in \text{Ker}(\xi)$ . Conversely, if for certain  $B \in u^*(\mathcal{H})$  we have  $\langle Bx, y \rangle_{\mathcal{H}} = 0$  for all  $x, y \in \text{Ker}(\xi)$ , then  $B$  can be written in the form  $v\xi + \xi v^\dagger$ . To see this, consider the splitting  $\mathcal{H} = V_1 \oplus V_2$ , where  $V_2 = \text{Ker}(\xi)$  and  $V_1 = V_2^\perp$ . According to this splitting  $\xi$  can be written in the operator matrix form

$$\xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\xi_1$  is Hermitian and invertible. Similarly,  $B$  has the form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix},$$

where  $B_{11}^\dagger = B_{11}$  and, in the obvious sense,  $B_{21} = B_{12}^\dagger$ . Now, it is easy to see that  $B = v\xi + \xi v^\dagger$ , where

$$v = \begin{pmatrix} \frac{1}{2} B_{11} \xi_1^{-1} & \xi_1 B_{12} \\ B_{21} \xi_1^{-1} & 0 \end{pmatrix}$$

that proves (41).

Finally, if  $u_{k_+,k_-}^*(\mathcal{H})$  intersects  $\mathcal{P}(\mathcal{H})$ , then it contains an element with non-negative spectrum. But the signs of the elements of the spectrum are constant along a  $GL(\mathcal{H})$ -orbit which means that  $k_- = 0$  and  $u_{k_+,k_-}^*(\mathcal{H}) = \mathcal{P}^k(\mathcal{H}) \subset \mathcal{P}(\mathcal{H})$ .  $\square$

Note that the fundamental vector fields  $\tilde{a}(\xi) = -a\xi - \xi a^\dagger$  of the  $GL(\mathcal{H})$ -action satisfy the commutation rules  $[\tilde{a}, \tilde{b}]_{vf} = [\tilde{a}, \tilde{b}]_-$ .

The next result shows that the foliation into submanifolds  $u_{k_+,k_-}^*(\mathcal{H})$  can be obtained directly from tensors  $\Lambda$  and  $R$ . We know already that the (generalized) distribution  $D_\Lambda$  induced by  $\Lambda$  is generated by vector fields  $\Lambda_A(\xi) = \# \Lambda_\xi(A) = [A, \xi]$  and the generalized distribution

$D_R$  induced by  $R$  is generated by vector fields  $R_A(\xi) = \#R_\xi(\widehat{A}) = [A, \xi]_+$ . The following is straightforward.

**Theorem 6.** *The family  $\{\Lambda_A, R_A : A \in u^*(\mathcal{H})\}$  of linear vector fields on  $u^*(\mathcal{H})$  is the family of fundamental vector fields of the  $GL(\mathcal{H})$ -action:*

$$\Lambda_A(\xi) = \frac{1}{i}(A\xi - \xi A) = -(iA)\xi - \xi(iA)^\dagger = i\widetilde{A}(\xi), \quad (42)$$

$$R_A(\xi) = A\xi + \xi A = A\xi + \xi A^\dagger = -\widetilde{A}(\xi). \quad (43)$$

In particular,

$$[\Lambda_A, \Lambda_B]_{vf} = \Lambda_{[A, B]}, \quad [R_A, R_B]_{vf} = \Lambda_{[A, B]}, \quad [R_A, \Lambda_B]_{vf} = R_{[A, B]}, \quad (44)$$

so the (generalized) distribution induced jointly by the tensors  $\Lambda$  and  $R$  is completely integrable and  $u_{k_+, k_-}^*(\mathcal{H})$  are the maximal integrate submanifolds.

**Corollary 4.** *The generalized distributions  $D_{gl} = D_R + D_\Lambda$ ,  $D_\Lambda$  and  $D_0 = D_R \cap D_\Lambda$  on  $u^*(\mathcal{H})$  are involutive and can be integrated to generalized foliations  $\mathcal{F}_{gl}$ ,  $\mathcal{F}_\Lambda$  and  $\mathcal{F}_0$ , respectively. The leaves of the foliation  $\mathcal{F}_{gl}$  are the orbits of the  $GL(\mathcal{H})$  action  $\xi \mapsto T\xi T^\dagger$ , the leaves of  $\mathcal{F}_\Lambda$  are the orbits of the  $U(\mathcal{H})$ -action.*

Denote by  $\widetilde{\mathcal{J}}$  and  $\widetilde{\mathcal{R}}$  the  $(1, 1)$ -tensors on  $u^*(\mathcal{H})$ , viewed as a vector bundle morphism induced by the contravariant tensors  $\Lambda$  and  $R$ , respectively,

$$\widetilde{\mathcal{J}}, \widetilde{\mathcal{R}} : Tu^*(\mathcal{H}) \rightarrow Tu^*(\mathcal{H}),$$

$$\widetilde{\mathcal{J}}_\xi(A) = [A, \xi] = \Lambda_\xi(A),$$

$$\widetilde{\mathcal{R}}_\xi(A) = [A, \xi]_+ = R_\xi(A),$$

where  $A \in u^*(\mathcal{H}) \simeq T_\xi u^*(\mathcal{H})$ . The image of  $\widetilde{\mathcal{J}}$  is  $D_\Lambda$  and the image of  $\widetilde{\mathcal{R}}$  is  $D_R$ .

**Lemma 4.** *The tensors  $\widetilde{\mathcal{J}}$  and  $\widetilde{\mathcal{R}}$  commute and*

$$\widetilde{\mathcal{J}}_\xi \circ \widetilde{\mathcal{R}}_\xi(A) = \widetilde{\mathcal{R}}_\xi \circ \widetilde{\mathcal{J}}_\xi(A) = [A, \xi^2]. \quad (45)$$

**Proof.** We have

$$\widetilde{\mathcal{J}}_\xi \circ \mathcal{R}_\xi(A) = [[A, \xi]_+, \xi].$$

But, as easily seen,

$$[[A, \xi]_+, \xi] = [A, \xi^2] = [[A, \xi], \xi]_+ = \widetilde{\mathcal{R}}_\xi \circ \widetilde{\mathcal{J}}_\xi(A). \quad (46)$$

□

Recall that  $U(\mathcal{H})$ -orbits  $\mathcal{O}$ , i.e. the orbits with respect to the action of the subgroup  $U(\mathcal{H}) \subset GL(\mathcal{H})$ , carry canonical symplectic structures  $\eta^\mathcal{O}$ . The symplectic structure  $\eta^\mathcal{O}$  is  $U(\mathcal{H})$ -invariant, i.e.  $(\mathcal{O}, \eta^\mathcal{O})$  is a homogeneous symplectic manifold. We will show that this symplectic structure is a part of a canonical Kähler structure. We already know this structure for the orbits  $\mathcal{P}_r^1(\mathcal{H})$ .

Recall also that on  $u^*(\mathcal{H})$  we have the Riemannian metric induced by the scalar product  $\langle A, B \rangle_{u^*} = \frac{1}{2} \text{Tr}(AB)$  on  $u^*(\mathcal{H})$ .

**Theorem 7.**

- (a) *The image of  $\widetilde{\mathcal{J}}_\xi$  is  $T_\xi \mathcal{O}$  and  $\text{Ker}(\widetilde{\mathcal{J}}_\xi)$  is the orthogonal complement of  $T_\xi \mathcal{O}$ .*  
 (b)  *$\widetilde{\mathcal{J}}_\xi^2$  is a self-adjoint (with respect to  $\langle \cdot, \cdot \rangle_{u^*}$ ) and negatively defined operator on  $T_\xi \mathcal{O}$ .*

(c) The  $(1, 1)$ -tensor  $\mathcal{J}$  on  $u^*(\mathcal{H})$  defined by

$$\mathcal{J}_\xi(A) = \left(-(\tilde{\mathcal{J}}_\xi^2)_{|T_\xi \mathcal{O}}\right)^{-\frac{1}{2}} \tilde{\mathcal{J}}_\xi(A) \tag{47}$$

induces an  $U(\mathcal{H})$ -invariant complex structure  $\mathcal{J}$  on every orbit  $\mathcal{O}$ .

(d) The tensor

$$\gamma_\xi^\mathcal{O}(A, B) = \eta_\xi^\mathcal{O}(A, \mathcal{J}_\xi(B)) \tag{48}$$

is an  $U(\mathcal{H})$ -invariant Riemannian metric on  $\mathcal{O}$  and

$$\gamma_\xi^\mathcal{O}(\mathcal{J}_\xi(A), B) = \eta_\xi^\mathcal{O}(A, B). \tag{49}$$

In particular,  $(\mathcal{O}, \mathcal{J}, \eta^\mathcal{O}, \gamma^\mathcal{O})$  is a homogeneous Kähler manifold. Moreover, if  $\xi \in u^*(\mathcal{H})$  is a projector and  $\xi \in \mathcal{O}$ , then  $\mathcal{J}_\xi = \tilde{\mathcal{J}}_\xi$  and  $\gamma^\mathcal{O}(A, B) = \langle A, B \rangle_{u^*}$ .

**Remark.** The tensor  $\mathcal{J}$  is canonically and globally defined. It is however not smooth as a tensor field on  $u^*(\mathcal{H})$ . It is smooth on the open-dense subset of regular elements and, of course, on every  $U(\mathcal{H})$ -orbit separately.

**Proof.**

(a) The vector fields  $\Lambda_A(\xi) = [A, \xi] = \tilde{\mathcal{J}}_\xi(A)$  are fundamental vector fields of the  $U(\mathcal{H})$ -action, so  $T_\xi \mathcal{O}$  is the image of  $\tilde{\mathcal{J}}_\xi$ . Moreover, the invariance of the Riemannian metric  $\langle A, B \rangle_{u^*}$ ,

$$\langle \tilde{\mathcal{J}}_\xi(A), B \rangle_{u^*} = \langle [A, \xi], B \rangle_{u^*} = -\langle A, \tilde{\mathcal{J}}_\xi(B) \rangle_{u^*}, \tag{50}$$

implies that

$$B \in \text{Ker}(\tilde{\mathcal{J}}_\xi) \Leftrightarrow B \perp \tilde{\mathcal{J}}_\xi(u^*(\mathcal{H})).$$

(b) The identity (50) means that  $\tilde{\mathcal{J}}_\xi^\times = -\tilde{\mathcal{J}}_\xi$ , where  $\tilde{\mathcal{J}}_\xi^\times$  is the adjoint operator to  $\tilde{\mathcal{J}}_\xi$  with respect to the scalar product  $\langle A, B \rangle_{u^*}$ . Consequently,

$$(\tilde{\mathcal{J}}_\xi^2)^\times = \tilde{\mathcal{J}}_\xi^2. \tag{51}$$

Moreover,  $\tilde{\mathcal{J}}_\xi^2$  is negatively defined on  $T_\xi \mathcal{O}$ , since

$$\langle \tilde{\mathcal{J}}_\xi^2(A), A \rangle_{u^*} = \langle [[A, \xi], \xi], A \rangle_{u^*} = -\langle [A, \xi], [A, \xi] \rangle_{u^*} < 0,$$

for  $[A, \xi] \in T_\xi \mathcal{O}$ ,  $[A, \xi] \neq 0$ .

(c) The tensor  $\tilde{\mathcal{J}}$  is clearly  $U(\mathcal{H})$ -invariant:

$$\tilde{\mathcal{J}}_{U\xi U^\dagger}(UAU^\dagger) = [UAU^\dagger, U\xi U^\dagger] = U[A, \xi]U^\dagger = U(\tilde{\mathcal{J}}_\xi(A))U^\dagger, \tag{52}$$

so the tensor  $(-\tilde{\mathcal{J}}^2)^{-\frac{1}{2}}$  and its composition  $\mathcal{J}$  are  $U(\mathcal{H})$ -invariant. The tensor  $\mathcal{J}$  defines an almost complex structure on every orbit  $\mathcal{O}$ , since

$$[(-\tilde{\mathcal{J}}^2)^{-\frac{1}{2}} \tilde{\mathcal{J}}]^2 = (-\tilde{\mathcal{J}}^2)^{-1} \tilde{\mathcal{J}}^2 = -I.$$

To show that this almost complex structure is integrable, it is sufficient to show that the distribution  $\mathcal{N}$  in the complexified tangent bundle  $T\mathcal{O} \otimes \mathbf{C}$  which consists of  $i$ -eigenvectors of (complexified)  $\mathcal{J}$  is involutive. Since  $\mathcal{J}$ , and therefore  $\mathcal{N}$ , is invariant, it is sufficient to check it at one point, say  $\xi \in \mathcal{O}$ , with respect to the complexified Lie algebra  $gl(\mathcal{H}) = u^*(\mathcal{H}) \otimes \mathbf{C}$  equipped with the bracket  $[a, b] = \frac{1}{i}[ab - ba]$ .

Let  $-\kappa_1^2, \dots, -\kappa_m^2$ , where  $\kappa_1, \dots, \kappa_m > 0$ , be the eigenvalues of  $(\tilde{\mathcal{J}}_\xi^2)_{|T_\xi \mathcal{O}}$  counted with multiplicities. The complexified  $\tilde{\mathcal{J}}_\xi$ , which with some abuse of notation we will denote by the same symbol, has therefore eigenvalues  $\pm i\kappa_k$  with eigenvectors  $a_k^\pm, k = 1, \dots, m$  and  $\mathcal{J}_\xi(a_k^\pm) = \pm i a_k^\pm$ . Thus  $\mathcal{N}_\xi$  is spanned by the vectors  $a_k^+, k = 1, \dots, m$ , i.e. eigenvectors

of  $\tilde{\mathcal{J}}_\xi$ ,  $\tilde{\mathcal{J}}_\xi(a_k^+) = i\kappa_k a_k^+$  with positive  $\kappa_k$ . This space is clearly a Lie subalgebra in  $gl(\mathcal{H})$ , since

$$\begin{aligned}\mathcal{J}_\xi([a_k^+, a_l^+]) &= [[a_k^+, a_l^+], \xi] = [[a_k^+, \xi], a_l^+] + [a_k^+, [a_l^+, \xi]] \\ &= [i\kappa_k a_k^+, a_l^+] + [a_k^+, i\kappa_l a_l^+] = i(\kappa_k + \kappa_l)[a_k^+, a_l^+],\end{aligned}$$

the vector  $[a_k^+, a_l^+]$ , if non-zero, is again an eigenvector of  $\tilde{\mathcal{J}}_\xi$  corresponding to a ‘positive’ eigenvalue  $i(\kappa_k + \kappa_l)$ .

(d) The tensor

$$\gamma_\xi^\mathcal{O}(A, B) = \eta_\xi^\mathcal{O}(A, \mathcal{J}_\xi(B))$$

is clearly  $U(\mathcal{H})$ -invariant. From (50) and (51) it follows that  $\mathcal{J}_\xi^\times = -\mathcal{J}_\xi$ . Since  $\tilde{\mathcal{J}}$  and  $\mathcal{J}$  clearly commute,  $\mathcal{J}_\xi([A, \xi]) = [\mathcal{J}_\xi(A), \xi]$ , in view of (17),

$$\begin{aligned}\eta_\xi^\mathcal{O}([A, \xi], \mathcal{J}_\xi([B, \xi])) &= \langle [A, \xi], \mathcal{J}_\xi(B) \rangle_{u^*(\mathcal{H})} = \langle -\mathcal{J}_\xi([A, \xi]), B \rangle_{u^*(\mathcal{H})} \\ &= -\eta_\xi^\mathcal{O}(\mathcal{J}_\xi([A, \xi]), [B, \xi]).\end{aligned}\quad (53)$$

This immediately implies that  $\gamma^\mathcal{O}$  is symmetric and proves (49). But (17) implies also that

$$\begin{aligned}\gamma_\xi^\mathcal{O}([A, \xi], [A, \xi]) &= \eta_\xi^\mathcal{O}([A, \xi], \mathcal{J}_\xi([A, \xi])) = \langle [A, \xi], \mathcal{J}_\xi(A) \rangle_{u^*(\mathcal{H})} \\ &= \langle A, -\tilde{\mathcal{J}}_\xi \mathcal{J}_\xi(A) \rangle_{u^*(\mathcal{H})}.\end{aligned}\quad (54)$$

But

$$-\tilde{\mathcal{J}}_\xi \mathcal{J}_\xi = (-\tilde{\mathcal{J}}^2)^{\frac{1}{2}}$$

is a positive operator, so

$$\gamma_\xi^\mathcal{O}([A, \xi], [A, \xi]) > 0$$

for  $[A, \xi] \neq 0$ .

Finally, if  $\xi$  is a projector,  $\xi^2 = \xi$ , then (cf (22))

$$\tilde{\mathcal{J}}_\xi^2([A, \xi]) = -[A, \xi],$$

so  $\mathcal{J}_\xi = \tilde{\mathcal{J}}_\xi$  and (cf (54))

$$\gamma_\xi^\mathcal{O}([A, \xi], [B, \xi]) = \langle [A, \xi], \mathcal{J}_\xi(B) \rangle_{u^*(\mathcal{H})} = \langle [A, \xi], [B, \xi] \rangle_{u^*(\mathcal{H})}.$$

□

We have some similar results for the tensor  $\tilde{\mathcal{R}}$  which however are not completely analogous, since the distribution  $D_R$  is not globally integrable. The proofs are analogous, so we omit them.

### Theorem 8.

- (a) The image  $D_R(\xi)$  of  $\tilde{\mathcal{R}}_\xi$  is the orthogonal complement of  $\text{Ker}(\tilde{\mathcal{R}}_\xi)$ .  
 (b)  $\tilde{\mathcal{R}}_\xi^2$  is a self-adjoint (with respect to  $\langle \cdot, \cdot \rangle_{u^*}$ ) and a positively defined operator on  $D_R(\xi)$ .  
 (c) The  $(1, 1)$ -tensor  $\mathcal{R}$  on  $u^*(\mathcal{H})$  defined by

$$\mathcal{R}_\xi(A) = |(\tilde{\mathcal{R}}_\xi)_{|D_R(\xi)}|^{-1} \circ \tilde{\mathcal{R}}_\xi(A) \quad (55)$$

satisfies  $\mathcal{R}^3 = \mathcal{R}$ .

**Corollary 5.** *The distribution  $D_0$  is the image of  $\mathcal{J}_\xi \circ \mathcal{R}_\xi = \mathcal{R}_\xi \circ \mathcal{J}_\xi$ . In other words,  $D_0(\xi) = \{[A, \xi^2] : A \in u^*(\mathcal{H})\}$ . Moreover, the foliation  $\mathcal{F}_0$  is  $U(\mathcal{H})$ -invariant,  $\mathcal{J}$ -invariant and  $\mathcal{R}$ -invariant, so that  $\mathcal{J}$  and  $\mathcal{R}$  induce on leaves of  $\mathcal{F}_0$  a complex and a product structure, respectively. The leaves of the foliation  $\mathcal{F}_0$  are also canonically symplectic manifolds with symplectic structures being restrictions of symplectic structures on the leaves of  $\mathcal{F}_\Lambda$ , so the leaves of  $\mathcal{F}_0$  are Kähler submanifolds of the  $U(\mathcal{H})$ -orbits in  $u^*(\mathcal{H})$ .*

**Proof.** The image of  $\mathcal{J}_\xi \circ \mathcal{R}_\xi = \mathcal{R}_\xi \circ \mathcal{J}_\xi$  is clearly contained in  $D_0$ . Conversely, let  $B \in D_0(\xi) = D_\Lambda \cap D_{\mathbf{R}}$ . According to (46),  $D_0(\xi)$  is invariant with respect to both:  $\mathcal{J}_\xi$  and  $\mathcal{R}_\xi$  and  $\mathcal{J}_\xi$  and  $\mathcal{R}_\xi$  are injective, thus surjective, on  $D_0(\xi)$ . The distribution  $D_0$  is therefore generated by vector fields  $X_A(\xi) = [A, \xi^2]$ . It is a matter of simple calculations to show that these vector fields commute with the fundamental vector fields  $\Lambda_B$  of the  $U(\mathcal{H})$  as  $[X_A, \Lambda_B]_{vf} = X_{[B, A]}$  that shows  $U(\mathcal{H})$  invariance of  $D_0$ . One can also easily see that the restrictions of  $\xi^\mathcal{O}$  to the leaves of  $\mathcal{F}_0$  contained in  $\mathcal{O}$  are non-degenerate. It follows also directly from the explicit calculations we present below.  $\square$

Let us explain the above theorem in local coordinates, i.e. for the case of matrices. Suppose that  $\xi = \text{diag}(\lambda_1, \dots, \lambda_n) \in u^*(n)$  is a diagonal matrix. For simplicity, it is better to start already with the complexified structures; i.e. with  $gl(n) = u^*(n) \otimes \mathbf{C}$  equipped with the bracket  $[a, b] = \frac{1}{i}(ab - ba)$  and the Hermitian product  $\langle a, b \rangle_{gl} = \frac{1}{2} \text{Tr}(a^\dagger b)$ , so that  $u^*(n)$  is a real Lie subalgebra in  $gl(n)$  with the induced scalar product. Let  $E_l^k$  be the matrix whose only non-zero entry is 1 at the  $k$ th row and  $l$ th column. We have

$$\langle E_l^k, E_s^r \rangle_{gl} = \frac{1}{2}(\delta_k^r \delta_s^l), \tag{56}$$

$$[E_l^k, E_s^r] = -i(\delta_l^r E_s^k - \delta_s^k E_l^r), \tag{57}$$

and

$$[E_l^k, E_s^r]_+ = \delta_l^r E_s^k + \delta_s^k E_l^r, \tag{58}$$

so that

$$\tilde{\mathcal{J}}_\xi(E_l^k) = [E_l^k, \xi] = i(\lambda_k - \lambda_l)E_l^k. \tag{59}$$

and

$$\tilde{\mathcal{R}}_\xi(E_l^k) = [E_l^k, \xi]_+ = (\lambda_k + \lambda_l)E_l^k. \tag{60}$$

In particular,

$$\tilde{\mathcal{J}}_\xi \circ \tilde{\mathcal{R}}_\xi(E_l^k) = [E_l^k, \xi^2] = i(\lambda_k^2 - \lambda_l^2)E_l^k. \tag{61}$$

Consequently,

$$\tilde{\mathcal{J}}_\xi^2(E_l^k) = -(\lambda_k - \lambda_l)^2 E_l^k \tag{62}$$

and

$$\tilde{\mathcal{R}}_\xi^2(E_l^k) = (\lambda_k + \lambda_l)^2 E_l^k, \tag{63}$$

so that

$$\mathcal{J}_\xi(E_l^k) = i \cdot \text{sgn}(\lambda_k - \lambda_l)E_l^k \tag{64}$$

and

$$\mathcal{R}_\xi(E_l^k) = \text{sgn}(\lambda_k + \lambda_l)E_l^k. \tag{65}$$

The (complexified) tangent space  $T_\xi \mathcal{O} \otimes \mathbf{C}$  is spanned by those  $E_l^k$  for which  $\lambda_k - \lambda_l \neq 0$ , the space  $D_{\mathbf{R}}(\xi) \otimes \mathbf{C}$  is spanned by those  $E_l^k$  for which  $\lambda_k + \lambda_l \neq 0$ , the space  $D_0(\xi) \otimes \mathbf{C}$  is

spanned by those  $E_l^k$  for which  $\lambda_k^2 - \lambda_l^2 \neq 0$ , and the distribution  $\mathcal{N}$  mentioned in the proof of the theorem is spanned by  $E_l^k$  for which  $\lambda_k - \lambda_l > 0$ . The complexified symplectic form reads

$$\eta_\xi^\mathcal{O}(\mathbf{i}(\lambda_k - \lambda_l)E_l^k, \mathbf{i}(\lambda_r - \lambda_s)E_s^r) = \langle \mathbf{i}(\lambda_k - \lambda_l)E_l^k, E_s^r \rangle_{gl} = -\mathbf{i}(\lambda_k - \lambda_l)\frac{1}{2}(\delta_l^r \delta_s^k),$$

i.e.

$$\eta_\xi^\mathcal{O}(E_l^k, E_s^r) = \frac{1}{2\mathbf{i}(\lambda_r - \lambda_s)}(\delta_l^r \delta_s^k), \quad (66)$$

and the complexified Riemannian form

$$\gamma_\xi^\mathcal{O}(E_l^k, E_s^r) = \eta_\xi^\mathcal{O}(E_l^k, \mathcal{J}_\xi(E_s^r)) = \frac{1}{2|\lambda_r - \lambda_s|}(\delta_l^r \delta_s^k). \quad (67)$$

As a basis in  $u^*(n)$  let us take

$$A_l^k = E_l^k + E_k^l, \quad k \leq l, \quad B_l^k = \mathbf{i}E_l^k - \mathbf{i}E_k^l, \quad k < l. \quad (68)$$

It is easy to see that this is an orthonormal basis and that

$$\mathcal{J}_\xi(A_l^k) = \operatorname{sgn}(\lambda_k - \lambda_l)B_l^k, \quad \mathcal{J}_\xi(B_l^k) = \operatorname{sgn}(\lambda_l - \lambda_k)A_l^k \quad (69)$$

and

$$\mathcal{R}_\xi(A_l^k) = \operatorname{sgn}(\lambda_k + \lambda_l)A_l^k, \quad \mathcal{J}_\xi(B_l^k) = \operatorname{sgn}(\lambda_l + \lambda_k)B_l^k. \quad (70)$$

Moreover,

$$\eta_\xi^\mathcal{O}(B_l^k, A_s^r) = \frac{\delta_l^k \delta_s^r}{(\lambda_k - \lambda_l)}, \quad \eta_\xi^\mathcal{O}(B_l^k, B_s^r) = \eta_\xi^\mathcal{O}(A_l^k, A_s^r) = 0, \quad \lambda_k - \lambda_l, \lambda_r - \lambda_s \neq 0 \quad (71)$$

and

$$\gamma_\xi^\mathcal{O}(B_l^k, A_s^r) = 0, \quad \gamma_\xi^\mathcal{O}(B_l^k, B_s^r) = \gamma_\xi^\mathcal{O}(A_l^k, A_s^r) = \frac{\delta_l^k \delta_s^r}{|\lambda_k - \lambda_l|}, \quad \lambda_k - \lambda_l, \lambda_r - \lambda_s \neq 0. \quad (72)$$

In other words

$$\eta_\xi^\mathcal{O} = \sum_{\lambda_k - \lambda_l \neq 0} \frac{1}{(\lambda_k - \lambda_l)} \cdot db_l^k \wedge da_l^k, \quad (73)$$

and

$$\gamma_\xi^\mathcal{O} = \sum_{\lambda_k - \lambda_l \neq 0} \frac{1}{|\lambda_k - \lambda_l|} (db_l^k \otimes db_l^k + da_l^k \otimes da_l^k), \quad (74)$$

where

$$b_l^k = \langle B_l^k, \cdot \rangle_{u^*}, \quad a_l^k = \langle A_l^k, \cdot \rangle_{u^*}$$

are coordinates on  $u^*(n)$  such that  $B_l^k = \partial_{b_l^k}$ ,  $A_l^k = \partial_{a_l^k}$ . The reduction of the symplectic form  $\eta^\mathcal{O}$  to the leaves of the foliation  $\mathcal{F}_0$ ,

$$(\eta_\xi^\mathcal{O})|_{\mathcal{F}_0} = \sum_{\lambda_k^2 - \lambda_l^2 \neq 0} \frac{1}{(\lambda_k - \lambda_l)} \cdot db_l^k \wedge da_l^k, \quad (75)$$

is clearly non-degenerate and constitutes, together with the reduced Riemannian structure

$$(\gamma_\xi^\mathcal{O})|_{\mathcal{F}_0} = \sum_{\lambda_k^2 - \lambda_l^2 \neq 0} \frac{1}{|\lambda_k - \lambda_l|} (db_l^k \otimes db_l^k + da_l^k \otimes da_l^k), \quad (76)$$

a Kähler structure.

**Remark.** Of course, when  $\xi$  is a projector, then  $\lambda_k = 1, 0$ , so  $\lambda_k - \lambda_l \neq 0 \Rightarrow |\lambda_k - \lambda_l| = 1$  and  $\gamma_\xi^O$  reduces to the canonical scalar product. Note also that the leaves of  $\mathcal{F}_\Lambda$  and  $\mathcal{F}_0$  through  $\xi$  coincide, except for the rare case when there are  $\lambda, \lambda' \neq 0$  in the spectrum of  $\xi$  such that  $\lambda + \lambda' = 0$ . In particular, the foliations  $\mathcal{F}_\Lambda$  and  $\mathcal{F}_0$  coincide when reduced to the subset  $\mathcal{P}(\mathcal{H})$  of non-negative operators or to the set  $\mathcal{D}(\mathcal{H})$  of density states. On such leaves the product structure  $\mathcal{R}$  is trivially the identity.

### 6. Composite systems and separability

Suppose now that our Hilbert space has a fixed decomposition into the tensor product of two Hilbert spaces  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ . This additional input is crucial in studying composite quantum systems and it has a great impact on the geometrical structures we have considered. The rest of this paper will be devoted to related problems.

Observe first that the tensor product map

$$\otimes : \mathcal{H}^1 \times \mathcal{H}^2 \rightarrow \mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2 \tag{77}$$

associates the product of rays with a ray, so it induces a canonical imbedding on the level of complex projective spaces

$$\text{Seg} : P\mathcal{H}^1 \times P\mathcal{H}^2 \rightarrow P\mathcal{H} = P(\mathcal{H}^1 \otimes \mathcal{H}^2), \tag{78}$$

$$(|x^1\rangle\langle x^1|, |x^2\rangle\langle x^2|) \mapsto |x^1 \otimes x^2\rangle\langle x^1 \otimes x^2|. \tag{79}$$

This imbedding of product of complex projective spaces into the projective space of the tensor product is called in the literature the *Segre imbedding* [41]. The elements of the image  $\text{Seg}(P\mathcal{H}^1 \times P\mathcal{H}^2)$  in  $P\mathcal{H} = P(\mathcal{H}^1 \otimes \mathcal{H}^2)$  are called *separable* pure states (with respect to the decomposition  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ ).

The Segre imbedding is related to the (external) tensor product of the basic representations of the unitary groups  $U(\mathcal{H}^1)$  and  $U(\mathcal{H}^2)$ , i.e. with the representation of the direct product group in  $\mathcal{H} = \mathcal{H}^1 \otimes \mathcal{H}^2$ ,

$$U(\mathcal{H}^1) \times U(\mathcal{H}^2) \ni (\rho^1, \rho^2) \mapsto \rho^1 \otimes \rho^2 \in U(\mathcal{H}) = U(\mathcal{H}^1 \otimes \mathcal{H}^2),$$

$$(\rho^1 \otimes \rho^2)(x^1 \otimes x^2) = \rho^1(x^1) \otimes \rho^2(x^2).$$

Note that  $\rho^1 \otimes \rho^2$  is unitary, since the Hermitian product in  $\mathcal{H}$  is related to the Hermitian products in  $\mathcal{H}^1$  and  $\mathcal{H}^2$  by

$$\langle x^1 \otimes x^2, y^1 \otimes y^2 \rangle_{\mathcal{H}} = \langle x^1, y^1 \rangle_{\mathcal{H}^1} \cdot \langle x^2, y^2 \rangle_{\mathcal{H}^2}. \tag{80}$$

The above group imbedding which, with some abuse of notation, we will denote by

$$\text{Seg} : U(\mathcal{H}^1) \times U(\mathcal{H}^2) \rightarrow U(\mathcal{H})$$

gives rise to the corresponding imbedding of Lie algebras

$$\text{Seg} : u(\mathcal{H}^1) \times u(\mathcal{H}^2) \rightarrow u(\mathcal{H}),$$

or, by our identification, of their duals

$$\text{Seg} : u^*(\mathcal{H}^1) \times u^*(\mathcal{H}^2) \rightarrow u^*(\mathcal{H}). \tag{81}$$

The original Segre imbedding is just the latter map reduced to pure states. In fact, a more general result holds true.

**Proposition 5.** *The imbedding (81) maps  $\mathcal{P}^k(\mathcal{H}^1) \times \mathcal{P}^l(\mathcal{H}^2)$  into  $\mathcal{P}^{kl}(\mathcal{H}^1 \otimes \mathcal{H}^2)$  and  $\mathcal{D}^k(\mathcal{H}^1) \times \mathcal{D}^l(\mathcal{H}^2)$  into  $\mathcal{D}^{kl}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ .*



**Proof.** Let us take  $A^1 \in \mathcal{P}^k(\mathcal{H}^1)$  and  $A^2 \in \mathcal{P}^l(\mathcal{H}^2)$ . Using bases of eigenvectors  $(e_i^1)$  of  $A^1$  and  $(e_j^2)$  of  $A^2$  to construct a basis  $(e_i^1 \otimes e_j^2)$  of eigenvectors of  $A^1 \otimes A^2$ , one easily sees that the elements of the spectrum of  $A^1 \otimes A^2$  (counted with multiplicities) are  $\lambda_i \lambda'_j$ , where  $A^1(e_i^1) = \lambda_i e_i^1$  and  $A^2(e_j^2) = \lambda'_j e_j^2$ , so that  $A^1 \otimes A^2 = \text{Seg}(A^1, A^2)$  is non-negatively defined and has rank  $kl$ . If  $A^1, A^2$  have trace 1, i.e.  $\sum_i \lambda_i = 1$  and  $\sum_j \lambda'_j = 1$ , then  $\sum_{i,j} \lambda_i \lambda'_j = \sum_i \lambda_i \cdot \sum_j \lambda'_j = 1$ .  $\square$

Let us denote the image  $\text{Seg}(\mathcal{D}^k(\mathcal{H}^1) \times \mathcal{D}^l(\mathcal{H}^2))$  by  $\mathcal{S}^{k,l}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , the set  $\mathcal{S}^{1,1}(\mathcal{H}^1 \otimes \mathcal{H}^2)$  of separable pure states simply by  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , and the convex hull

$$\text{conv}(\text{Seg}(\mathcal{D}(\mathcal{H}^1) \times \mathcal{D}(\mathcal{H}^2)))$$

of the subset  $\text{Seg}(\mathcal{D}(\mathcal{H}^1) \times \mathcal{D}(\mathcal{H}^2))$  of all separable states in  $u^*(\mathcal{H})$  by  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . The states from

$$\mathcal{E}(\mathcal{H}^1 \otimes \mathcal{H}^2) = \mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2) \setminus \mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2),$$

i.e. those which are not separable, are called *entangled states*.

**Proposition 6.** *The convex set  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$  of separable states is the convex hull of the set  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$  of separable pure states and  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$  is exactly the set of extremal points of  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Moreover,  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , thus  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , is invariant with respect to the canonical  $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$ -action on  $u^*(\mathcal{H}^1 \otimes \mathcal{H}^2)$ ,*

$$(T_1, T_2)A = (T_1 \otimes T_2) \circ A \circ (T_1 \otimes T_2)^\dagger.$$

**Proof.** Let us start by showing that the convex hull of  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$  contains  $\text{Seg}(\mathcal{D}(\mathcal{H}^1) \times \mathcal{D}(\mathcal{H}^2))$  thus equals  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Indeed  $\mathcal{D}^1(\mathcal{H}^i)$  is the set of extreme points of  $\mathcal{D}(\mathcal{H}^i)$ ,  $i = 1, 2$ , so that any  $A^i \in \mathcal{D}(\mathcal{H}^i)$  is a convex combination  $A^i = t_s^i \rho_s^i$  of elements  $\rho_s^i \in \mathcal{D}^1(\mathcal{H}^i)$ ,  $i = 1, 2$ . Hence,  $A^1 \otimes A^2$  is the convex combination

$$A^1 \otimes A^2 = \sum_{s,s'} t_s^1 t_{s'}^2 \cdot \rho_s^1 \otimes \rho_{s'}^2.$$

On the other hand, every state  $\rho^1 \otimes \rho^2$ ,  $\rho^i \in \mathcal{D}(\mathcal{H}^i)$ ,  $i = 1, 2$ , is in  $\mathcal{D}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , i.e. it is an extremal point of  $\mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Therefore it cannot be written as a non-trivial convex combination of elements from  $\mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ , thus from a smaller set  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . The invariance is obvious, since

$$(T_1 \otimes T_2) \circ (\rho_1 \otimes \rho_2) \circ (T_1^\dagger \otimes T_2^\dagger) = (T_1 \rho_1 T_1^\dagger) \otimes (T_2 \rho_2 T_2^\dagger)$$

and  $(T_1 \rho_1 T_1^\dagger) \in \mathcal{D}^1(\mathcal{H}^1)$  for  $\rho_1 \in \mathcal{D}^1(\mathcal{H}^1)$ .  $\square$

Since we are working in a finite-dimensional space, the closeness of the corresponding hulls is automatic that can be derived from the following lemma.

**Lemma 5.** *If  $V$  is an  $n$ -dimensional real vector space and  $x$  is a convex combination  $x = \sum_{i=1}^m t_i x_i$  of certain points of  $V$ , then  $x$  is a convex combination of at most  $(n + 1)$  points among  $x_i$ 's.*

**Proof.** It suffices to prove that  $x$  is a convex combination of  $(m - 1)$  of  $x_i$ 's, provided  $m > n + 1$ . Of course, we can assume that all  $t_i > 0$ . If  $m > n + 1$ , then there are  $a_i \in \mathbf{R}$ , not all equal to 0, such that  $\sum_1^m a_i = 0$  and  $\sum_1^m a_i x_i = 0$ . There is  $i_0$  such that  $|a_{i_0}/t_{i_0}|$  is maximal among  $|a_i/t_i|$ ,  $i = 1, \dots, m$ . We can assume without loss of generality that  $i_0 = m$ . Hence

$$x = \sum_{i=1}^{m-1} \left( t_i - \frac{a_i t_m}{a_m} \right) x_i$$

and the above combination is convex, since  $(t_i - \frac{a_i t_m}{a_m}) \geq 0$  and

$$\sum_1^{m-1} \left( t_i - \frac{a_i t_m}{a_m} \right) = \sum_1^m \left( t_i - \frac{a_i t_m}{a_m} \right) = \sum_1^m t_i = 1. \quad \square$$

**Proposition 7.** *The convex hull  $\text{conv}(E)$  of a compact subset  $E$  in a finite-dimensional real vector space  $V$  is compact.*

**Proof.** Suppose that the dimension of the space is  $n$  and denote by  $\Delta_{n+1}$  the compact  $(n + 1)$ -dimensional simplex

$$\Delta_{n+1} = \left\{ t = (t_1, \dots, t_{n+1}) : t_i \geq 0, \sum_1^{n+1} t_i = 1 \right\}.$$

According to the above lemma,  $\text{conv}(E)$  is the image of the compact set  $\Delta \times E \times \dots \times E$  ( $E$  appears in the product  $(n + 1)$ -times) under the continuous map

$$\Delta \times E \times \dots \times E \ni (t, x_1, \dots, x_{n+1}) \mapsto \sum_1^{n+1} t_i x_i \in V. \quad \square$$

**Corollary 6.** *The set  $\mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$  is a compact subset of  $u^*(\mathcal{H}^1 \otimes \mathcal{H}^2)$ .*

The entangled states play an important role in quantum computing and one of the main problems is to decide effectively whether a given composite state is entangled or not. An abstract measurement of entanglement can be based on the following observation (see also [42]).

Let  $E$  be the set of all extreme points of a compact convex set  $K$  in a finite-dimensional real vector space  $V$  and let  $E_0$  be a compact subset of  $E$  with the convex hull  $K_0 = \text{conv}(E_0) \subset K$ . For every non-negative function  $f : E \rightarrow \mathbf{R}_+$  define its extension  $f_K : K \rightarrow \mathbf{R}_+$  by

$$f_K(x) = \inf_{x = \sum t_i \alpha_i} \sum t_i f(\alpha_i), \tag{82}$$

where the *infimum* is taken with respect to all expressions of  $x$  in the form of convex combinations of points from  $E$ . Recall that, according to Krein–Milman theorem,  $K$  is the convex hull of its extreme points.

**Theorem 9.** *For every non-negative continuous function  $f : E \rightarrow \mathbf{R}_+$  which vanishes exactly on  $E_0$  the function  $f_K$  is convex on  $K$  and vanishes exactly on  $K_0$*

**Proof.** It is completely obvious that  $f_K$  vanishes on the convex hull of  $E_0$ . The function  $f_K$  is convex, since for every convex combination  $x = t_i y_i$  of points of  $K$  and every  $\varepsilon > 0$  we can find extreme points  $\alpha_j$  with convex combinations  $y_i = s_i^j \alpha_j$  and  $f_K(y_i) > s_i^j f(\alpha_j) - \varepsilon$ . Hence

$$f_K(t_i y_i) = f_K(t_i s_i^j \alpha_j) \leq t_i s_i^j f(\alpha_j) < t_i (f(y_i) + \varepsilon) = t_i f_K(y_i) + \varepsilon.$$

Due to arbitrariness of  $\varepsilon > 0$  we get

$$f_K(t_i y_i) \leq t_i f_K(y_i).$$

Note finally that  $f_K$  vanishes exactly on  $K_0$ . Indeed  $K_0$  is compact due to proposition 7 and if  $x \notin K_0$ , then  $x$  and  $K_0$  can be separated by a hyperplane; i.e. there is a linear functional  $\phi : V \rightarrow \mathbf{R}$  such that  $\phi(x) = a > 0$  and  $\phi$  is negative on  $K_0$ . Denote by  $E_1$  the (compact) set of those points from  $E$  on which  $\phi$  takes non-negative values and by  $F$  the minimum of  $f$  on  $E_1$ .

Of course,  $F > 0$ , since  $E_1 \cap E_0 = \emptyset$ . Let  $M \in \mathbf{R}$  be the maximum of  $\phi$  on  $E$ . Of course,  $M > 0$ . For any realization  $x = t_i \alpha_i$  of  $x$  as a convex combination of points of  $E$  we have

$$a = \phi(x) = \sum_i t_i \phi(\alpha_i) \leq \sum_{\alpha_i \in E_1} t_i \phi(\alpha_i) \leq M \sum_{\alpha_i \in E_1} t_i.$$

On the other hand,

$$\sum_i t_i f(\alpha_i) \geq \sum_{\alpha_i \in E_1} t_i f(\alpha_i) \geq F \sum_{\alpha_i \in E_1} t_i \geq \frac{aF}{M},$$

so  $f_K(x) \geq \frac{aF}{M} > 0$ . □

**Corollary 7.** Let  $F : \mathcal{D}^1(\mathcal{H}^1 \otimes \mathcal{H}^2) \rightarrow \mathbf{R}_+$  be a continuous function which vanishes exactly on  $\mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Then

$$\mu = F_{\mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2)} : \mathcal{D}(\mathcal{H}^1 \otimes \mathcal{H}^2) \rightarrow \mathbf{R}_+$$

is a measure of entanglement, i.e.  $\mu$  is convex and  $\mu(x) = 0 \Leftrightarrow x \in \mathcal{S}(\mathcal{H}^1 \otimes \mathcal{H}^2)$ . Moreover, if  $f$  is taken  $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$ -invariant, then  $\mu$  is  $U(\mathcal{H}^1) \times U(\mathcal{H}^2)$ -invariant.

**Proof.** The first part is a direct consequence of theorem 9. Also the invariance of  $\mu$  is clear:

$$\mu(T\rho T^\dagger) = \inf_W (t_i f(\alpha_i)) = \inf_{W'} (t_i f(T\alpha_i T^\dagger)) = \inf_{W'} (t_i f(\alpha_i)) = \mu(\rho),$$

where  $T$  is in the corresponding group,

$$W = \left\{ (t_i, \alpha_i) : T\rho T^\dagger = \sum t_i \alpha_i, \alpha_i \in \mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2), t_i \geq 0, \sum t_i = 1 \right\},$$

and

$$W' = \left\{ (t_i, \alpha_i) : \rho = \sum t_i \alpha_i, \alpha_i \in \mathcal{S}^1(\mathcal{H}^1 \otimes \mathcal{H}^2), t_i \geq 0, \sum t_i = 1 \right\}. \quad \square$$

A careful study of the geometry of  $u^*(\mathcal{H}^1 \otimes \mathcal{H}^2)$  and criteria of entanglement we postpone to a separate paper.

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